Scale Invariance with non metric measures, Inflation, Dark Energy with No 5th Force, etc.

Eduardo Guendelman, Ben Gurion University,Israel.

Talk at the CANTATA 4th CONFERENCE, TUZLA, October 9th, 2019.

We introduce 4 scalar fields in 4-D

Which will play an important geometrical role, by the way, other authors have found that 4 scalar fields in 4-D can be used to define a generally covariant mass term for the graviton . A simpler use of four scalars in 4D is to define a new MEASURE. Can also define this measure from the curl of a 3 index field, but we will work with the 4 scalars.

The Basic Idea of the Two Measures Theory (TMT)

The general structure of general coordinate invariant theories is taken usually as

$$S_1 = \int L_1 \sqrt{-g}\, d^4x\,, \tag{1}$$

where $g = \det(g_{\mu\nu})$. The introduction of $\sqrt{-g}$ is required since d^4x by itself is not a scalar but the product $\sqrt{-g}\, d^4x$ is a scalar. Inserting $\sqrt{-g}$, which has the transformation properties of a density, produces a scalar action S_1, as defined by Eq. (1), provided L_1 is a scalar.

In principle, nothing prevents us from considering other densities instead of $\sqrt{-g}$. One construction of such alternative "measure of integration," is obtained as follows: given 4-scalars φ_a $(a = 1, 2, 3, 4)$, one can construct the density

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d \tag{2}$$

One can consider both contributions, and allowing therefore both geometrical objects to enter the theory and take as our action

$$S = \int L_1 \sqrt{-g}\, d^4x + \int L_2 \Phi\, d^4x\,. \tag{6}$$

Here L_1 and L_2 are φ_a independent. There is a good reason not to consider nonlinear terms in Φ that mix Φ with $\sqrt{-g}$, for example

$$\frac{\Phi^2}{\sqrt{-g}} \tag{7}$$

appear.

This is because S in Eq. (6) is invariant (up to the integral of a total divergence) under the infinite-dimensional symmetry

$$\varphi_a \to \varphi_a + f_a(L_2)\,, \tag{8}$$

where $f_a(L_2)$ is an arbitrary function of L_2 if L_1 and L_2 are φ_a independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms (like (7)) are present. Therefore (6) is considered for the case when no dependence on the measure fields (MF) appears in L_1 or L_2.

Softly Broken Conformal Invariance, simple example

$$S_L = \int L_1 \sqrt{-g}\, d^4x + \int L_2 \Phi\, d^4x \,, \tag{9}$$

$$L_1 = U(\phi) \,, \tag{10}$$

$$L_2 = \frac{-1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \,, \tag{11}$$

$$R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \,, \quad R_{\mu\nu}(\Gamma) = R^{\lambda}_{\mu\nu\lambda} \,, \tag{12}$$

$$R^{\lambda}_{\mu\nu\sigma}(\Gamma) = \Gamma^{\lambda}_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma,\nu} + \Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} \,. \tag{13}$$

In the variational principle $\Gamma^{\lambda}_{\mu\nu}, g_{\mu\nu}$, the measure fields scalars φ_a and the "matter"-scalar field ϕ are all to be treated as independent variables although the variational principle may result in equations that allow us to solve some of these variables in terms of others.

For the case where the potential terms $U = V = 0$, we have local conformal invariance

$$g_{\mu\nu} \to \Omega(x) g_{\mu\nu} \tag{14}$$

and φ_a is transformed according to

$$\varphi_a \to \varphi'_a = \varphi'_a(\varphi_b), \tag{15}$$

$$\Phi \to \Phi' = J(x)\Phi, \tag{16}$$

where $J(x)$ is the Jacobian of the transformation of the φ_a fields.

This will be a symmetry in the case $U = V = 0$ if

$$\Omega = J. \tag{17}$$

Notice that J can be a local function of space–time, this can be arranged by performing for the φ_a fields one of the (infinite) possible diffeomorphism in the internal φ_a space.

Let us study now the equations obtained from the variation of the connections $\Gamma^{\lambda}_{\mu\nu}$. We obtain then

$$-\Gamma^{\lambda}_{\mu\nu}-\Gamma^{\alpha}_{\beta\mu}g^{\beta\lambda}g_{\alpha\nu}+\delta^{\lambda}_{\nu}\Gamma^{\alpha}_{\mu\alpha}+\delta^{\lambda}_{\mu}g^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta}g_{\gamma\nu}-g_{\alpha\nu}\partial_{\mu}g^{\alpha\lambda}+\delta^{\lambda}_{\mu}g_{\alpha\nu}\partial_{\beta}g^{\alpha\beta}-\delta^{\lambda}_{\nu}\frac{\Phi_{,\mu}}{\Phi}+\delta^{\lambda}_{\mu}\frac{\Phi_{,\nu}}{\Phi}=O \quad (22)$$

If we define $\Sigma^{\lambda}_{\mu\nu}$ as $\Sigma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}-\{{\lambda \atop \mu\nu}\}$ where $\{{\lambda \atop \mu\nu}\}$ is the Christoffel symbol, we obtain for $\Sigma^{\lambda}_{\mu\nu}$ the equation

$$-\sigma_{,\lambda}\,g_{\mu\nu}+\sigma_{,\mu}\,g_{\nu\lambda}-g_{\nu\alpha}\Sigma^{\alpha}_{\lambda\mu}-g_{\mu\alpha}\Sigma^{\alpha}_{\nu\lambda}+g_{\mu\nu}\Sigma^{\alpha}_{\lambda\alpha}+g_{\nu\lambda}g_{\alpha\mu}g^{\beta\gamma}\Sigma^{\alpha}_{\beta\gamma}=O \quad (23)$$

where $\sigma=1n\chi, \chi=\frac{\Phi}{\sqrt{-g}}$.

The general solution of (23) is

$$\Sigma^{\alpha}_{\mu\nu}=\delta^{\alpha}_{\mu}\lambda_{,\nu}+\frac{1}{2}(\sigma_{,\mu}\,\delta^{\alpha}_{\nu}-\sigma_{,\beta}\,g_{\mu\nu}g^{\alpha\beta}) \quad (24)$$

where λ is an arbitrary function due to the λ - symmetry of the curvature[5] $R^{\lambda}_{\mu\nu\alpha}(\Gamma)$,

$$\Gamma^{\alpha}_{\mu\nu}\rightarrow\Gamma'^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{\mu\nu}+\delta^{\alpha}_{\mu}Z_{,\nu} \quad (25)$$

Z being any scalar (which means $\lambda\rightarrow\lambda+Z$).

If we choose the gauge $\lambda=\frac{\sigma}{2}$, we obtain

$$\Sigma^{\alpha}_{\mu\nu}(\sigma)=\frac{1}{2}(\delta^{\alpha}_{\mu}\sigma_{,\nu}+\delta^{\alpha}_{\nu}\sigma_{,\mu}-\sigma_{,\beta}\,g_{\mu\nu}g^{\alpha\beta}). \quad (26)$$

special exponential form for the U and V potentials. Indeed, if we perform the global scale transformation ($\theta = \text{const}$)

$$g_{\mu\nu} \to e^{\theta} g_{\mu\nu} \,, \tag{18}$$

then (9) is invariant provided $V(\phi)$ and $U(\phi)$ are of the form[25]

$$V(\phi) = f_1 e^{\alpha\phi} \,, \quad U(\phi) = f_2 e^{2\alpha\phi} \tag{19}$$

and φ_a is transformed according to

$$\varphi_a \to \lambda_{ab} \varphi_b \,, \tag{20}$$

which means

$$\Phi \to \det(\lambda_{ab}) \Phi \equiv \lambda \Phi \tag{21}$$

such that

$$\lambda = e^{\theta} \tag{22}$$

and

$$\phi \to \phi - \frac{\theta}{\alpha} \,. \tag{23}$$

4. Spontaneously Broken Scale Invariance

Now we will solve for the scalar

$$\chi = \frac{\Phi}{\sqrt{-g}}$$

$$A^\mu_a \partial_\mu L_2 = 0\,, \tag{24}$$

where $A^\mu_a = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d$. Since it is easy to check that $A^\mu_a\partial_\mu\varphi_{a'} = \frac{\delta aa'}{4}\Phi$, it follows that $\det(A^\mu_a) = \frac{4^{-4}}{4!}\Phi^3 \neq 0$ if $\Phi \neq 0$. Therefore if $\Phi \neq 0$ we obtain that $\partial_\mu L_2 = 0$, or that

$$L_2 = \frac{-1}{\kappa}R(\Gamma, g) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V = M\,, \tag{25}$$

Considering now the variation with respect to $g^{\mu\nu}$, we obtain

$$\Phi\left(\frac{-1}{\kappa}R_{\mu\nu}(\Gamma) + \frac{1}{2}\phi_{,\mu}\phi_{,\nu}\right) - \frac{1}{2}\sqrt{-g}U(\phi)g_{\mu\nu} = 0\,. \tag{31}$$

Solving for $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ from Eq. (31) and introducing in Eq. (25), we obtain

$$M + V(\phi) - \frac{2U(\phi)}{\chi} = 0\,, \tag{32}$$

a constraint that allows us to solve for χ,

$$\chi = \frac{2U(\phi)}{M + V(\phi)}\,. \tag{33}$$

To get the physical content of the theory, it is best consider variables that have well-defined dynamical interpretation. The original metric does not has a nonzero canonical momenta. The fundamental variable of the theory in the first-order formalism is the connection and its canonical momenta is a function of $\bar{g}_{\mu\nu}$ given by

$$\bar{g}_{\mu\nu} = \chi g_{\mu\nu} \tag{34}$$

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$\overline{g}_{\mu\nu} = \chi g_{\mu\nu} \tag{30}$$

and χ given by (29b). In terms of $\overline{g}_{\mu\nu}$ the non Riemannian contribution $\Sigma^{\alpha}_{\mu\nu}$ dissappears from the equations, which can be written then in the Einstein form ($R_{\mu\nu}(\overline{g}_{\alpha\beta})$ = usual Ricci tensor) and χ given by Eq. (33). Interestingly enough, working with $\bar{g}_{\mu\nu}$ is the same as going to the "Einstein conformal frame." In terms of $\bar{g}_{\mu\nu}$ the non-Riemannian contribution $\Sigma^{\alpha}_{\mu\nu}$ disappears from the equations. This is because the connection can be written as the Christoffel symbol of the metric $\bar{g}_{\mu\nu}$. In terms of $\bar{g}_{\mu\nu}$, the equations of motion for the metric can be written then in the Einstein form (we define $\bar{R}_{\mu\nu}(\bar{g}_{\alpha\beta})$ = usual Ricci tensor in terms of the bar metric $= R_{\mu\nu}$ and $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$)

$$\bar{R}_{\mu\nu}(\bar{g}_{\alpha\beta}) - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}(\bar{g}_{\alpha\beta}) = \frac{\kappa}{2}T^{\text{eff}}_{\mu\nu}(\phi)\,, \tag{35}$$

where

$$T^{\text{eff}}_{\mu\nu}(\phi) = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\phi_{,\alpha}\phi_{,\beta}\bar{g}^{\alpha\beta} + \bar{g}_{\mu\nu}V_{\text{eff}}(\phi) \tag{36}$$

and

$$V_{\text{eff}}(\phi) = \frac{1}{4U(\phi)}(V+M)^2\,. \tag{37}$$

In terms of the metric $\bar{g}^{\alpha\beta}$, the equation of motion of the scalar field ϕ takes the standard general-relativity form

$$\frac{1}{\sqrt{-\bar{g}}}\partial_\mu\left(\bar{g}^{\mu\nu}\sqrt{-\bar{g}}\partial_\nu\phi\right) + V'_{\text{eff}}(\phi) = 0\,. \tag{38}$$

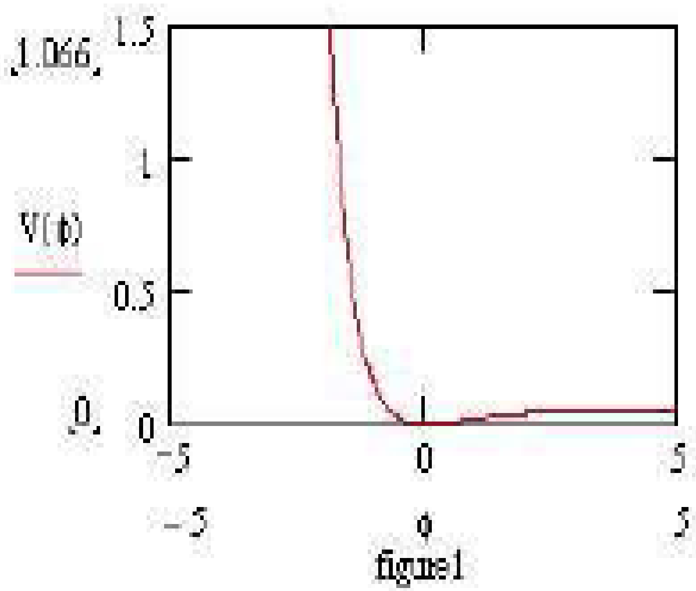
Notice that if $V + M = 0$, $V_{\text{eff}} = 0$ and $V'_{\text{eff}} = 0$ also, provided V' is finite and $U \neq 0$ there. This means the zero cosmological constant state is achieved without any sort of fine-tuning. That is, independently of whether we add to V a constant

Effective Potential for Exponential forms of U and V (scale invariance)

$$V(\phi) := \frac{\left(e^{\phi} - 1\right)^2}{20 e^{2\phi}}$$

$M < 0$

$\varepsilon := 0$



The 2nd order formulation

A similar looking action, but where we use the second order formulation leads to very different results, in particular the ratio of the two measures is not determined by a constrained, rather

It becomes a new degree of freedom that could play the role of a curvaton , see

Two scalar fields inflation from scale-invariant gravity with modified measure
David Benisty , Eduardo I. Guendelman
Published in **Class.Quant.Grav. 36 (2019) no.9, 095001**
DOI: 10.1088/1361-6382/ab14af
e-Print: **arXiv:1809.09866 [gr-qc]**

Dynamically Generated Inflation from Non-Riemannian Volume Forms
David Benisty, Eduardo Guendelman, Emil Nissimov, Svetlana Pacheva
Published in **Eur.Phys.J. C79 (2019) no.9, 806**
DOI: 10.1140/epjc/s10052-019-7310-6
e-Print: **arXiv:1906.06691 [gr-qc] |**

Related developments and things that I will not cover in this talk

A related development to the Two MeasuresTheory (TMT) is the recently studied “Lagrange multiplier gravity” (LMG), where a Lagrange multiplier field is introduced so as to force some lagrangian to be zero, instead of a constant. These two approaches are very much related, just that for the “dictionary”, we must incorporate in the LMG the constant of integration M into the Lagrangian that is being forced to vanish. TMT predates LMG. A very rich subject that I will not talk on here concerns the use of TMT or LMG for obtaining dark matter in addition to dark energy. See for example papers with Singleton and Yomgram and more recently with Nissimov and Pacheva.

UNIMODULAR THEORY IN TMT FORM:

.

$$S = \int d^4x \Phi L_1 + \int d^4x \sqrt{-g} L_2$$

TMT form (13) is studied, where $L_1 = \frac{2\Lambda}{16\pi G}$ and $L_2 = -\frac{1}{16\pi G}(R + 2\Lambda)$, in the notation of [6] $\Phi = \partial_\mu T^\mu$, here Λ is taken as a dynamical variable, but the equation (as in any TMT type theory) $L_1 = M = constant$, forces Λ to be a constant, the variation of Λ, gives a relation between the two measures, $\Phi = \partial_\mu T^\mu = \sqrt{-g}$. We see that of course the

6. Generation of Two Flat Regions after the Introduction of a R^2 Term

As we have seen, it is possible to obtain a model that through a spontaneous breaking of scale invariance can give us a flat region. We want to obtain now two flat regions in our effective potential. A simple generalization of the action S_L will fix this. What one needs to do is simply consider the addition of a scale invariant term of the form

$$S_{R^2} = \epsilon \int (g^{\mu\nu} R_{\mu\nu}(\Gamma))^2 \sqrt{-g}\, d^4x \,. \tag{45}$$

The total action being then $S = S_L + S_{R^2}$.[38] In the first-order formalism, S_{R^2} is not only globally scale invariant but also locally scale invariant, that is conformally

The variation of the action with respect to $g^{\mu\nu}$ gives now

$$R_{\mu\nu}(\Gamma)\left(\frac{-\Phi}{\kappa} + 2\epsilon R\sqrt{-g}\right) + \Phi\frac{1}{2}\phi_{,\mu}\,\phi_{,\nu} - \frac{1}{2}\left(\epsilon R^2 + U(\phi)\right)\sqrt{-g}g_{\mu\nu} = 0\,. \tag{46}$$

It is interesting to notice that if we contract this equation with $g^{\mu\nu}$, the ϵ terms do not contribute. This means that the same value for the scalar curvature R is obtained as in Sec. 2, if we express our result in terms of ϕ, its derivatives and $g^{\mu\nu}$. Solving the scalar curvature from this and inserting in the other ϵ-independent equation $L_2 = M$, we get still the same solution for the ratio of the measures which was found in the case where the ϵ terms were absent, i.e. $\chi = \frac{\Phi}{\sqrt{-g}} = \frac{2U(\phi)}{M+V(\phi)}$.

metric $\bar{g}_{\mu\nu}$ given by

$$\bar{g}_{\mu\nu} = \left(\frac{\Omega}{\sqrt{-g}}\right) g_{\mu\nu} = (\chi - 2\kappa\epsilon R) g_{\mu\nu}\,, \tag{47}$$

$\bar{g}_{\mu\nu}$ defines now the "Einstein frame." Equations (46) can now be expressed in the "Einstein form"

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{\kappa}{2}T^{\text{eff}}_{\mu\nu}\,, \tag{48}$$

where

$$T^{\text{eff}}_{\mu\nu} = \frac{\chi}{\chi - 2\kappa\epsilon R}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\phi_{,\alpha}\phi_{,\beta}\bar{g}^{\alpha\beta}\right) + \bar{g}_{\mu\nu}V_{\text{eff}}\,, \tag{49}$$

$$V_{\text{eff}} = \frac{\epsilon R^2 + U}{(\chi - 2\kappa\epsilon R)^2}\,. \tag{50}$$

Here it is satisfied that $\frac{-1}{\kappa}R(\Gamma, g) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V = M$, equation that expressed in terms of $\bar{g}^{\alpha\beta}$ becomes $\frac{-1}{\kappa}R(\Gamma, g) + (\chi - 2\kappa\epsilon R)\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V = M$. This allows us to solve for R and we get

$$R = \frac{-\kappa(V + M) + \frac{\kappa}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\chi}{1 + \kappa^2\epsilon\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}\,. \tag{51}$$

Notice that if we express R in terms of ϕ, its derivatives and $g^{\mu\nu}$, the result is the same as in the model without the curvature squared term, this is not true anymore once we express R in terms of ϕ, its derivatives and $\bar{g}^{\mu\nu}$.

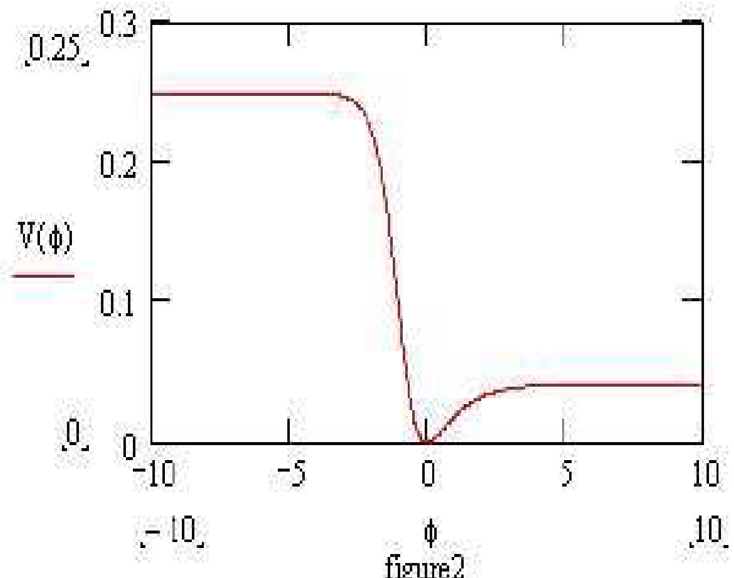
In any case, once we insert (51) into (50), we see that the effective potential (50) will depend on the derivatives of the scalar field now. It acts as a normal scalar field potential under the conditions of slow rolling or low gradients and in the case the scalar field is near the region $M + V(\phi) = 0$.

Potential with 2 flat regions M<0

$$V(\phi) := \frac{\left(e^{\phi} - 1\right)^2}{4\left[\left(e^{\phi} - 1\right)^2 + 5e^{2\phi}\right]}$$

$M < 0$

$\varepsilon > 0$



For M>0, we can get a “quintessential inflation” potential

$$V(\phi) := \frac{\left(e^{\phi}+1\right)^2}{4\left[\left(e^{\phi}+1\right)^2+5\,e^{2\phi}\right]}$$

$M > 0.$

$\varepsilon > 0$

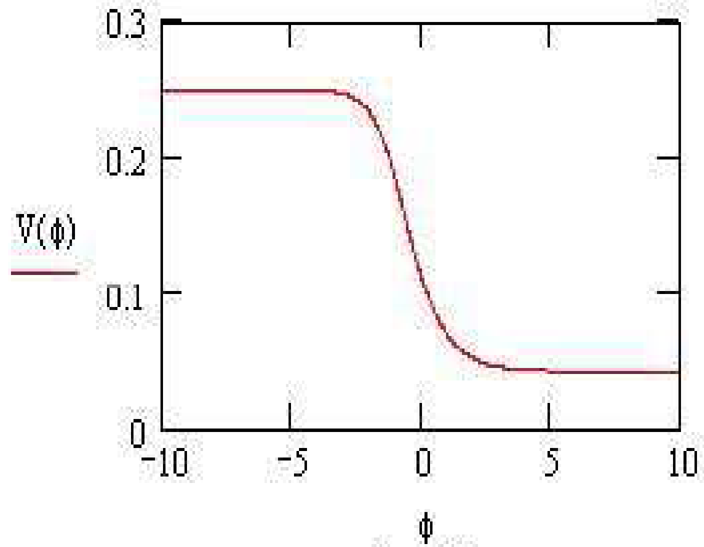


figure3

Second, for asymptotically large but negative values of the scalar field, we have

$$V_{\text{eff}}(\phi \to -\infty) \to \frac{1}{4\epsilon\kappa^2}\,. \qquad (54)$$

- Another flat region is also obtained for the other extreme, the dilaton field goes to plus infinity. If the constant of integration M is negative, there is a local minimum at zero for the effective potential, the value of the vacuum energy density in this región is 0.25 $\dfrac{f_1^2/f_2}{(1+\epsilon f_1^2/f_2)}$
- One can then

get a very small vacuum enery if f_1<<f_2.

in more details the Einstein frame

we see that the original metric does not have a canonically conjugated momentum (this turns out to be zero), in contrast, the canonically conjugated momentum to the conection turns out to be a function exclusively of $\overline{g}_{\mu\nu}$, this Einstein metric is therefore a genuine dynamical canonical variable, as opposed to the original metric. There is also a lagrangian formulation of the theory which uses $\overline{g}_{\mu\nu}$, as we will see in the next section, what we can call the action in the Einstein frame. In this frame we can quantize the theory for example and consider contributions without reference to the original frame, thus possibly considering breking the TMT structure of the theory through quantum effects, but such breaking will be done "softly" through the introduction of a cosmological term only. Surpringly, the remaining structure of the theory, reminiscent from the original TMT structure will be enough to control the strength of this additional cosmological term once we demand that the universe originated from a non singular and stable emergent state.

SCALE INVARIANCE AND AVOIDANCE OF 5^{th} FORCE

Recent papers on the subject, much latter than our contributions (but do not cite us),

No fifth force in a scale invariant universe

Pedro G. Ferreira, Christopher T. Hill, Graham G. Ross, **Phys.Rev. D95 (2017) no.6, 064038**

DOI: 10.1103/PhysRevD.95.064038

e-Print: **arXiv:1612.03157 [gr-qc]**

Our studies of avoidance of 5th force scale with invariance

Absence of the Fifth Force Problem in a Model with Spontaneously Broken Dilatation Symmetry E.I. Guendelman, A.B. Kaganovich
Published in **Annals Phys. 323 (2008) 866-882**

under the global scale transformations:

$$g_{\mu\nu} \to e^{\theta} g_{\mu\nu}, \quad \Gamma^{\mu}_{\alpha\beta} \to \Gamma^{\mu}_{\alpha\beta}, \quad \phi \to \phi - \frac{M_p}{\alpha}\theta, \quad \varphi_a \to l_{ab}\varphi_b \tag{7}$$

where $\det(l_{ab}) = e^{2\theta}$ and $\theta = const.$ Keeping the general structure (2), it is convenient to represent the action in the following form:

$$\begin{aligned}
S &= S_g + S_\phi + S_m \qquad (8)\\
S_g &= -\frac{1}{\kappa}\int (\Phi + b_g\sqrt{-g})R(\Gamma, g)e^{\alpha\phi/M_p}d^4x\,;\\
S_\phi &= \int e^{\alpha\phi/M_p}\left[(\Phi + b_\phi\sqrt{-g})\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \left(\Phi V_1 + \sqrt{-g}V_2\right)e^{\alpha\phi/M_p}\right]d^4x\,;\\
S_m &= \int (\Phi + b_m\sqrt{-g})L_m d^4x\,,
\end{aligned}$$

where

$$R(\Gamma, g) = g^{\mu\nu}\left(\Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}\right)$$

and the Lagrangian for the matter, as collection of particles, which provides the scale invariance of S_m reads

$$L_m = -m\sum_i \int e^{\frac{1}{2}\alpha\phi/M_p}\sqrt{g_{\alpha\beta}\frac{dx_i^\alpha}{d\lambda}\frac{dx_i^\beta}{d\lambda}}\frac{\delta^4(x - x_i(\lambda))}{\sqrt{-g}}d\lambda \tag{9}$$

where λ is an arbitrary parameter. For simplicity we consider the collection of the particles with the same mass parameter m. We assume in addition that $x_i(\lambda)$ do not participate in the scale transformations (7).

In the action (8) there are two types of the gravitational terms and of the "kinetic-like terms" which respect the scale invariance : the terms of the one type coupled to the measure Φ and those of the other type coupled to the measure $\sqrt{-g}$.

normalization of the measure fields φ_a we set the coupling constant of the scalar curvature to the measure Φ to be $-\frac{1}{\kappa}$. Normalizing all the fields such that their couplings to the measure Φ have no additional factors, we are not able in general to provide the same in terms describing the appropriate couplings to the measure $\sqrt{-g}$. This fact explains the need to introduce the dimensionless real parameters b_g, b_ϕ and b_m. We will only assume that they are positive, have the same or very close orders of magnitude

$$b_g \sim b_\phi \sim b_m \tag{10}$$

and besides $b_m > b_g$. The real positive parameter α is assumed to be of the order of unity. As usual $\kappa = 16\pi G$ and we use $M_p = (8\pi G)^{-1/2}$.

One should also point out the possibility of introducing two different pre-potentials which are exponential functions of the dilaton ϕ coupled to the measures Φ and $\sqrt{-g}$ with factors V_1 and V_2. Such ϕ-dependence provides the scale symmetry (7). We will see below how the dilaton effective potential is generated as the result of SSB of the scale invariance and the transformation to the Einstein frame.

According to the general prescriptions of TMT, we have to start from studying the self-consistent system of gravity (metric $g_{\mu\nu}$ and connection $\Gamma^{\mu}_{\alpha\beta}$), the measure Φ degrees of freedom φ_a, the dilaton field ϕ and the matter particles coordinates $x_i^{\alpha}(\lambda)$, proceeding in the first order formalism.

For the purpose of this paper we restrict ourselves to a zero temperature gas of particles, i.e. we will assume that $d\vec{x}_i/d\lambda \equiv 0$ for all particles. It is convenient to proceed in the frame where $g_{0l} = 0, \quad l = 1, 2, 3$. Then the particle density is defined by

$$n(\vec{x}) = \sum_i \frac{1}{\sqrt{-g_{(3)}}} \delta^{(3)}(\vec{x} - \vec{x}_i(\lambda)) \tag{11}$$

where $g_{(3)} = \det(g_{kl})$ and

$$S_m = -m \int d^4x (\Phi + b_m \sqrt{-g})\, n(\vec{x})\, e^{\frac{1}{2}\alpha\phi/M_p} \tag{12}$$

Following the procedure described in the previous subsection we have to write down all equations of motion, find the consistency condition (the constraint which determines ζ-field as a function of other fields and matter) and make a transformation to the Einstein frame.

the ratio of the two measures

$$\zeta \equiv \frac{\Phi}{\sqrt{-g}}$$

We will skip most of the intermediate results and in the next subsection present the resulting equations in the Einstein frame. Nevertheless two exclusions we have to make here.

The first one concerns the important effect observable when varying S_m with respect to $g^{\mu\nu}$:

$$\frac{\delta S_m}{\delta g^{00}} = \frac{b_m}{2}\sqrt{-g}\, m\, n(\vec{x})\, e^{\frac{1}{2}\alpha\phi/M_p}\, g_{00}, \tag{13}$$

$$\frac{\delta S_m}{\delta g^{kl}} = -\frac{1}{2}\Phi\, m\, n(\vec{x})\, e^{\frac{1}{2}\alpha\phi/M_p}\, g_{kl}. \tag{14}$$

The latter equation shows that due to the measure Φ, *the zero temperature gas generically possesses a pressure*. As we will see this pressure disappears automatically together with the fifth force as the matter energy density is many orders of magnitude larger then the dark energy density, which is evidently true in all physical phenomena tested experimentally.

The second one is the notion concerning the role of Eq. (5) resulting from variation of the measure fields φ_a. With the action (8), where the Lagrangian L_1 is the sum of terms coupled to the measure Φ, Eq. (5) describes a *spontaneous breakdown of the global scale symmetry* (7).

III. EQUATIONS OF MOTION IN THE EINSTEIN FRAME.

It turns out that when working with the new metric (ϕ remains the same)

$$\tilde{g}_{\mu\nu} = e^{\alpha\phi/M_p}(\zeta + b_g)g_{\mu\nu}, \tag{15}$$

which we call the Einstein frame, the connection becomes Riemannian. Since $\tilde{g}_{\mu\nu}$ is invariant under the scale transformations (7), spontaneous breaking of the scale symmetry (by means of Eq.(5)) is reduced in the Einstein frame to the *spontaneous breakdown of the shift symmetry* (1). Notice that the Goldstone theorem generically is not applicable in this kind of models[37].

The transformation (15) causes the transformation of the particle density

$$\tilde{n}(\vec{x}) = (\zeta + b_g)^{-3/2}\, e^{-\frac{3}{2}\alpha\phi/M_p}\, n(\vec{x}) \tag{16}$$

After the change of variables to the Einstein frame (15) and some simple algebra, the gravitational equations take the standard GR form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2}T^{eff}_{\mu\nu} \tag{17}$$

where $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$. The components of the effective energy-momentum tensor are as follows

$$T_{00}^{eff} = \frac{\zeta + b_\phi}{\zeta + b_g}\left(\dot{\phi}^2 - \tilde{g}_{00}X\right) + \tilde{g}_{00}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\zeta + b_g}X + \frac{3\zeta + b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right] \tag{18}$$

$$T_{ij}^{eff} = \frac{\zeta + b_\phi}{\zeta + b_g}\left(\phi_{,k}\phi_{,l} - \tilde{g}_{kl}X\right) + \tilde{g}_{kl}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\zeta + b_g}X + \frac{\zeta - b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right] \tag{19}$$

Here the following notations have been used:

$$X \equiv \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad and \qquad \delta = \frac{b_g - b_\phi}{b_g} \tag{20}$$

and the function $V_{eff}(\phi;\zeta)$ is defined by

$$V_{eff}(\phi;\zeta) = \frac{b_g\left[M^4 e^{-2\alpha\phi/M_p} + V_1\right] - V_2}{(\zeta + b_g)^2} \tag{21}$$

The dilaton ϕ field equation in the Einstein frame is as follows

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\zeta+b_\phi}{\zeta+b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right]$$
$$-\frac{\alpha}{M_p}\frac{(\zeta+b_g)M^4e^{-2\alpha\phi/M_p}-(\zeta-b_g)V_1-2V_2-\delta b_g(\zeta+b_g)X}{(\zeta+b_g)^2}$$
$$=\frac{\alpha}{M_p}\frac{\zeta-b_m+2b_g}{2\sqrt{\zeta+b_g}}m\,\tilde{n} \tag{22}$$

In the above equations, the scalar field ζ is determined as a function $\zeta(\phi, X, \tilde{n})$ by means of the following constraint (origin of which has been discussed in Sec.2.1):

$$\frac{(b_g-\zeta)\left(M^4e^{-2\alpha\phi/M_p}+V_1\right)-2V_2}{(\zeta+b_g)^2}-\frac{\delta\cdot b_gX}{\zeta+b_g}=\frac{\zeta-b_m+2b_g}{2\sqrt{\zeta+b_g}}m\,\tilde{n} \tag{23}$$

Applying the constraint (23) to Eq.(22) one can reduce the latter to the form

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\zeta+b_\phi}{\zeta+b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right]-\frac{2\alpha\zeta}{(\zeta+b_g)^2M_p}M^4e^{-2\alpha\phi/M_p}=0, \tag{24}$$

where ζ is a solution of the constraint (23).

One should point out two very important features of the model. First, the ϕ dependence in all the equations of motion (including the constraint) emerges only in the form $M^4 e^{-2\alpha\phi/M_p}$ where M is the integration constant, i.e. due to the spontaneous breakdown of the scale symmetry (7) (or the shift symmetry (1) in the Einstein frame). Second, the constraint (23) is the fifth degree algebraic equation with respect to $\sqrt{\zeta + b_g}$ and therefore generically ζ is a complicated function of ϕ, X and $\tilde{n}$. Hence generically each of ζ dependent terms in Eqs. (18)-(22) and (24) describe very nontrivial coupling of the dilaton to the matter.

IV. DARK ENERGY IN THE ABSENCE OF MATTER

It is worth to start the investigation of the features of our model from the simplest case when the particle density of the dust is zero: $\tilde{n}(x) \equiv 0$. Then the dilaton ϕ is the only matter which in the early universe plays the role of the inflaton while in the late universe it is the dark energy. The appropriate model in the context of cosmological solutions has been studied in detail in Ref. [41]. Here we present only some of the equations we will need for the purposes of this paper and a list of the main results.

In the absence of the matter particles, the scalar $\zeta = \zeta(\phi, X)$ can be easily found from the constraint (23):

$$\zeta^{(\tilde{n}=0)} = b_g - 2\frac{V_2 + \delta \cdot b_g^2 X}{M^4 e^{-2\alpha\phi/M_p} + V_1 + \delta \cdot b_a X} \qquad (25)$$

In the spatially homogeneous case $X \geq 0$. Then the effective energy-momentum tensor can be represented in a form of that of a perfect fluid

$$T^{eff}_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\tilde{g}_{\mu\nu}, \qquad where \qquad u_\mu = \frac{\phi_{,\mu}}{(2X)^{1/2}} \tag{26}$$

with the following energy and pressure densities obtained after inserting (25) into the components of the energy-momentum tensor (18), (19) where now $\tilde{n}(x) \equiv 0$

$$\rho(\phi, X; M) \equiv \rho^{(\tilde{n}=0)} \tag{27}$$
$$= X + \frac{(M^4 e^{-2\alpha\phi/M_p} + V_1)^2 - 2\delta b_g (M^4 e^{-2\alpha\phi/M_p} + V_1)X - 3\delta^2 b_g^2 X^2}{4[b_g(M^4 e^{-2\alpha\phi/M_p} + V_1) - V_2]},$$

$$p(\phi, X; M) \equiv p^{(\tilde{n}=0)} = X - \frac{\left(M^4 e^{-2\alpha\phi/M_p} + V_1 + \delta b_g X\right)^2}{4[b_g(M^4 e^{-2\alpha\phi/M_p} + V_1) - V_2]}. \tag{28}$$

Substitution of (25) into Eq. (24) yields the ϕ-equation with very interesting dynamics. The appearance of the nonlinear X dependence in spite of the absence of such nonlinearity in the underlying action means that our model represents an explicit example of k-essence[16] resulting from first principles. The effective k-essence action has the form

$$S_{eff} = \int \sqrt{-\tilde{g}} d^4x \left[-\frac{1}{\kappa} R(\tilde{g}) + p(\phi, X; M)\right], \tag{29}$$

where $p(\phi, X; M)$ is given by Eq.(28).

In the context of spatially flat FRW cosmology, in the absence of the matter particles (i.e $\tilde{n}(x) \equiv 0$), the TMT model under consideration[41] exhibits a number of interesting outputs depending of the choice of regions in the parameter space (but without fine tuning):

a) *Absence of initial singularity of the curvature while its time derivative is singular.* This is a sort of "sudden" singularities studied by Barrow on purely kinematic grounds[44].

b) Power law inflation in the subsequent stage of evolution. Depending on the region in the parameter space the inflation ends with a *graceful exit* either into the state with zero cosmological constant (CC) or into the state driven by both a small CC and the field ϕ with a quintessence-like potential.

c) Possibility of *resolution of the old CC problem.* From the point of view of TMT, it becomes clear why the old CC problem cannot be solved (without fine tuning) in conventional field theories.

d) TMT enables two ways for achieving small CC without fine tuning of dimensionful parameters: either by a *seesaw* type mechanism or due to *a correspondence principle* between TMT and conventional field theories (i.e theories with only the measure of integration $\sqrt{-g}$ in the action).

e) There is a wide range of the parameters where the dynamics of the scalar field ϕ, playing the role of the dark energy in the late universe, allows crossing the phantom divide, i.e. the equation-of-state $w = p/\rho$ may be $w < -1$ and w asymptotically (as $t \to \infty$) approaches -1 from below. One can show that in the original frame used in the underlying action (8), this regime corresponds to the negative sign of the measure of integration $\Phi + b_\phi\sqrt{-g}$ of the dilaton ϕ kinetic term[51]. This dynamical effect which emerges here instead of putting the wrong sign kinetic term by hand in the phantom model[46], will be discussed in detail in another paper.

Taking into account that in the late time universe the X-contribution to $\rho^{(\tilde{n}=0)}$ approaches zero, one can see that the dark energy density is positive for any ϕ provided

$$b_g V_1 \geq V_2 \tag{30}$$

Then it follows from (25) that

$$|\zeta^{(\tilde{n}=0)}| \sim b_g.$$

This will be useful in the next section.

V. NORMAL CONDITIONS: REPRODUCING EINSTEIN'S GR AND ABSENCE OF THE FIFTH FORCE PROBLEM

One should now pay attention to the interesting result that the explicit $\tilde{n}$ dependence involving **the same form of ζ dependence**

$$\frac{\zeta - b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n} \tag{32}$$

appears simultaneously[52] in the dust contribution to the pressure (through the last term in Eq. (19)), in the effective dilaton to dust coupling (in the r.h.s. of Eq. (22)) and in the r.h.s. of the constraint (23).

Let us analyze consequences of this wonderful coincidence in the case when the matter energy density (modeled by dust) is much larger than the dilaton contribution to the dark energy density in the space region occupied by this matter. Evidently this is the condition under which all tests of Einstein's GR, including the question of the fifth force, are fulfilled.

Therefore if this condition is satisfied we will say that the matter is in **normal conditions**. The existence of the fifth force turns into a problem just in normal conditions. The opposite situation may be realized (see Refs. [31],[32]) if the matter is diluted up to a magnitude of the macroscopic energy density comparable with the dilaton contribution to the dark energy density. In this case we say that the matter is in the state of cosmo-low energy physics (**CLEP**). It is evident that the fifth force acting on the matter in the CLEP state cannot be detected now and in the near future, and therefore does not appear to be a problem. But effects of the CLEP may be important in cosmology, see Ref. [32].

The last terms in eqs. (18) and (19), being the matter contributions to the energy density (ρ_m) and the pressure ($-p_m$) respectively, generally speaking have the same order of magnitude. But if the dust is in the normal conditions there is a possibility to provide the desirable feature of the dust in GR: it must be pressureless. This is realized provided that in normal conditions (n.c.) the following equality holds with extremely high accuracy:

$$\zeta^{(n.c.)} \approx b_m - 2b_g \tag{33}$$

Remind that we have assumed $b_m > b_g$. Then $\zeta^{(n.c.)} + b_g > 0$, and the transformation (15) and the subsequent equations in the Einstein frame are well defined. Inserting (33) in the last term of Eq. (18) we obtain the effective dust energy density in normal conditions

$$\rho_m^{(n.c.)} = 2\sqrt{b_m - b_g}\, m\tilde{n} \tag{34}$$

Substitution of (33) into the rest of the terms of the components of the energy-momentum tensor (18) and (19) gives the dilaton contribution to the energy density and pressure of the dark energy which have the orders of magnitude close to those in the absence of matter case, Eqs. (27) and (28). The latter statement may be easily checked by using Eqs. (25),(31),(33) and (10).

Note that Eq. (33) is not just a choice to provide zero dust contribution to the pressure. In fact it is the result of analyzing the equations of motion together with the constraint (23). In the Appendix we present the detailed analysis yielding this result. But in this section we have started from the use of this result in order to make the physical meaning more distinct.

Taking into account our assumption (10) and Eq. (31) we infer that $\zeta^{(n.c.)}$ and $\zeta^{(\tilde{n}=0)}$ (in the absence of matter case, Eq. (25)) have close orders of magnitudes. Then it is easy to see (making use the inequality (30)) that the l.h.s. of the constraint (23), as $\zeta = \zeta^{(n.c.)}$, has the order of magnitude close to that of the dark energy density $\rho^{(\tilde{n}=0)}$ in the absence of matter case discussed in Sec. 4. Thus in the case under consideration, *the constraint (23) describes a balance between the pressure of the dust in normal conditions on the one hand and the vacuum energy density on the other hand.* This balance is realized due to the condition (33).

Besides reproducing Einstein equations when the scalar field and dust (in normal conditions) are sources of the gravity, *the condition (33) automatically provides a practical disappearance of the effective dilaton to matter coupling.* Indeed, inserting (33) into the ϕ-equation written in the form (24) and into $V_{eff}(\phi;\zeta)$, Eq. (21), one can immediately see that only the force of the strength of the dark energy selfinteraction is present in this case. Note that this force is a total force involving both the selfinteraction of the dilaton and its interaction with dust in normal conditions. Furthermore, in this way one can see explicitly that due to the factor $M^4 e^{-2\alpha\phi/M_p}$, this total force may obtain an additional, exponential dumping since in the cosmological context shortly discussed in Sec. 4 (see details in Ref. ([41])) a scenario, where in the late time universe $\phi \gg M_p$, seems to be most appealing.

Another way to see the absence of the fifth force problem in the normal conditions is to look at the ϕ-equation in the form (22) and estimate the Yukawa type coupling constant in the r.h.s. of this equation. In fact, using the constraint (23) and representing the particle density in the form $\tilde{n} \approx N/\upsilon$ where N is the number of particles in a volume υ, one can make the following estimation for the effective dilaton to matter coupling "constant" f defined by the Yukawa type interaction term $f\tilde{n}\phi$ (if we were to invent an effective action whose variation with respect to ϕ would result in Eq. (22)):

$$f \equiv \alpha\frac{m}{M_p}\,\frac{\zeta - b_m + 2b_g}{2\sqrt{\zeta + b_g}} \approx \alpha\frac{m}{M_p}\,\frac{\zeta - b_m + 2b_g}{2\sqrt{b_m - b_g}} \sim \frac{\alpha}{M_p}\,\frac{\rho_{vac}}{\tilde{n}} \approx \alpha\frac{\rho_{vac}\upsilon}{NM_p} \tag{35}$$

Thus we conclude that *the effective coupling "constant" of the dilaton to matter in the normal conditions is of the order of the ratio of the "mass of the vacuum" in the volume occupied by the matter to the Planck mass taken N times*. In some sense this result resembles the *Archimedes law*. At the same time Eq. (35) gives us an estimation of the exactness of the

conclusions

Although the dust model studied in this paper is a very crude model of matter, it is quite sufficient for studying the fifth force problem. In fact, all experiments which search for the fifth force deal with macroscopic bodies which, in the zeroth order approximation, can be regarded as collections of noninteracting, point-like motionless particles with very high particle number density $\tilde{n}(x)$.

Generically the model studied in the present paper is different from Einstein's GR. For example it allows the long range scalar force and a non-zero pressure of the cold dust. However the magnitude of the particle number density turns out to be the very important factor influencing the strength of the dilaton to matter coupling. This happens due to the constraint (23) which is nothing but the consistency condition of the equations of motion. The analysis of the constraint presented in the Appendix shows that generically it describes a balance between the matter density and dark energy density. It turns out that in the case of a macroscopic body, that is in normal conditions, the constraint allows this balance only in such a way that the dilaton practically decouples from the matter and Einstein's GR is restored automatically. Thus our model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

Formally one can consider the case of a very diluted matter when the matter energy density is of the order of magnitude comparable with the dark energy density, which is the case opposite to the normal conditions. Only in this case the balance dictated by the constraint implies the existence of a non small dilaton coupling to matter, as well as a possibility of other distinctions from Einstein's GR. However these effects cannot be detected in fifth force experiments now and in the near future. One should also note here that in the framework of the present model based on the consideration of point particles, the low density limit, strictly speaking, cannot be satisfactory defined. An example of the appropriate low density limit (CLEP state) was realized using a field theory model in Ref. [32] while conclusions for matter in the normal conditions were very similar to results of the present paper.

Possible cosmological and astrophysical effects when the normal conditions are not satisfied may be very interesting. In particular, taking into account that all dark matter known in the present universe has the macroscopic energy density many orders of magnitude smaller than the energy density of visible macroscopic bodies, we hope that the nature of the dark matter can be understood as a state opposite to the normal conditions studied in the present

Some additional References

- **The** emergent **universe: Inflationary cosmology with no singularity.** G. F.R. Ellis, R.Maartens, **Class.Quant.Grav.21:223-232,2004**. e-Print: **gr-qc/0211082** .
- **Dynamical measure and field theory models …..**E.I. Guendelman, A.B.Kaganovich **Phys.Rev.D60:065004,1999**. e-Print: **gr-qc/9905029** .
- Model Constrained by S.I. : **Scale invariance, new inflation and** decaying **lambda terms.** **E.I. Guendelman**, **Mod.Phys.Lett.A14:1043-1052,1999**. e-Print: **gr-qc/9901017**
- Adding curvature Square and obtaining two flat regions : **Inflation and transition to a slowly accelerating phase from SSB of scale invariance.** E.I.Guendelman, O.Katz, **Class.Quant.Grav.20:1715-1728,2003**. e-Print: **gr-qc/0211095**
- **Using this model plus CC in Einstein frame to get ………Emerging Universe from Scale Invariance.** S. del Campo, E. I. Guendelman , R. Herrera , P.Labrana, **JCAP 1006:026,2010**. ePrint: **arXiv:1006.5734** [astro-ph.CO]
- **Expresing the results of the above in terms of the energy density of the flat region with lower energy density : Non Singular Origin of the Universe and its Present Vacuum Energy Density.** E.I. Guendelman, **Int.J.Mod.Phys.A26:2951-2972,2011**. e-Print: **arXiv:1103.1427** [gr-qc]