The why and how of Finsler field theory

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OUTLINE

- Motivation
- Positive definite Finsler geometry
- Definition of Finsler spacetimes
- The stage for Finsler field theories
- A concrete vacuum action
- Kinetic gas as a Finslerian field

1 "Who ordered Finsler?"

- Finsler geometry is a *framework*, allowing for many models;

- Just discarded 250 models out of 500? OK, then take another 500.

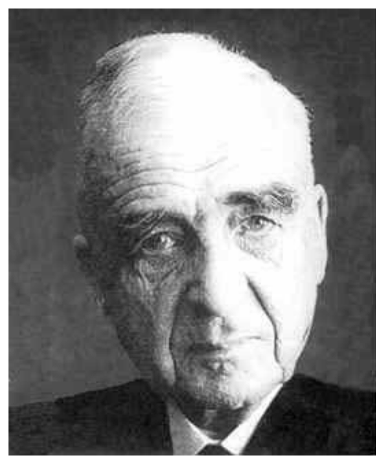
- **Still, why would we consider Finsler?**
  - modified dispersion relations, Lorentz violating standard model extensions;.
  - kinetic theory of gases - more accurate description of a gravitating gas;
  - address dark energy/dark matter geometrically

- **Pseudo-Finsler geometry** *includes* pseudo-Riemannian geometry;
  - Finsler field theories = extended field theories, GR is a particular case;
  - most general geometry with a *geometric clock*;
  - retains: weak equivalence principle, a precise notion of causal structure of spacetime;
  - generally, *no* local Lorentz invariance.

2 Positive definite Finsler geometry

- **Riemann (1854):** proposed: $ds = F(x, dx)$
($F \geq 0$, **homogeneous of the degree 1** and convex in dx).
- **1918: P. Finsler - first systematic study of such metrics** F

Bernhard Riemann
(1826–1866)



Paul Finsler
(1894-1970)

Take: M - smooth manifold, dim $M = 4$, $(x^i, \dot{x}^i)$ - coords on TM

Definition: $F : TM \to \mathbb{R}, \ (x, \dot{x}) \to F(x, \dot{x})$ - **Finsler norm**, if:

1) $F = F(x, \dot{x})$ is $\mathcal{C}^\infty$-differentiable for $\dot{x} \neq 0$;

2) $F(x, \alpha\dot{x}) = \alpha F(x, \dot{x})$, $\forall \alpha > 0$

3) The Hessian

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{x}^i \partial \dot{x}^j}$$

is *positive definite* for $\dot{x} \neq 0$.

Properties:

- each $F_x : T_x M \to \mathbb{R}$ - "norm" (not necessarily from a scalar product)
- rods and clocks - *yes;*
- but: no direct method of measuring angles

Length of a curve $c : t \in [a, b] \longrightarrow (x^i(t))$ in (M, F) :

$$l(c) = \int_a^b F(x(t), \frac{dx}{dt}(t))dt = \int_a^b \sqrt{g_{ij}(x, \dot{x})\dot{x}^i\dot{x}^j}dt.$$

Remark: $l(c)$ - *independent on the parametrization* of c.

Main classes of Finsler metrics:

1) *Riemann spaces*: $F(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$;

2) *Randers spaces:* $F(x, \dot{x}) = \sqrt{a_{ij}(x)\dot{x}^i\dot{x}^j} + b_i(x)\dot{x}^i$;

3) *Kropina spaces:* $F(x, \dot{x}) = a_{ij}(x)\dot{x}^i\dot{x}^j(b_k(x)\dot{x}^k)^{-1}$;

4) *m-th root metric spaces* $F(x, \dot{x}) = \sqrt[m]{a_{i_1 i_2 \ldots i_m}(x)\dot{x}^{i_1}\dot{x}^{i_2}\ldots\dot{x}^{i_m}}$.

Remark: In 3) and 4), F is NOT everywhere smooth for $\dot{x} \neq 0$ (!).

- Set: $\quad L = F^2 \quad \Rightarrow L(x, \dot{x}) = g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j.$

- Geodesics $s \mapsto (x^i(s))$ of a Finsler manifold:

$$\frac{d^2 x^i}{ds^2} + 2\Gamma^i{}_{jk}(x, \frac{dx}{ds})\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

In particular: if $g_{ij} = g_{ij}(x)$, then $2G^i(x, \dot{x}) = \Gamma^i{}_{jk}(x)\dot{x}^j \dot{x}^k$.

- Geodesic deviation, local behavior of neighboring geodesics – determined by *flag curvature*

$$R^i{}_j(x, \dot{x}) = R_k{}^i{}_{jl}(x, \dot{x})\dot{x}^k \dot{x}^l$$

($R^i{}_j$ - 2-homogeneous in the velocities).

3 Finsler spacetimes

Roughly speaking: (M, L) – **Finsler spacetime**, if:
- $L : TM \to \mathbb{R}$ - continuous everywhere, smooth *almost everywhere*;
- *good cone structure:*

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$$

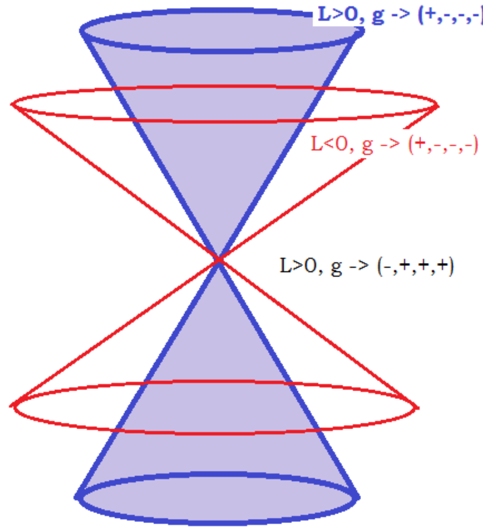
are defined, smooth and of Lorentzian signature for an entire connected component of a cone at each $x \in M$;
- local existence and uniqueness of *causal geodesics* with given initial conditions.

Interpretation:

$$ds^2 = L(x, dx)$$

- Definition below: M. Hohmann, C. Pfeifer, N. Voicu, PRD 100, 064035 (2019);

  We want to allow, e.g., for such things:



- $\mathcal{Q} \subset TM \backslash \{0\}$ - *conic subbundle* if: $(x, \dot{x}) \in \mathcal{Q} \Rightarrow (x, \alpha\dot{x}) \in \mathcal{Q}$, $\forall \alpha > 0$.

Finsler spacetime = a pair (M, L), where $L : TM \to \mathbb{R}$ is a continuous function, s.th.:

1) *Positive 2-homogeneity:* $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall \alpha > 0$.

2) *Almost everywhere smoothness:* $\exists$ a conic subbundle $\mathcal{A} \subset TM$ with $TM \backslash \mathcal{A}$ of measure zero, s.th., on $\mathcal{A}$, L is smooth and the matrix

$$g^L_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = \frac{1}{2} L_{\cdot i \cdot j}$$

is non-degenerate.

3) *Future directed timelike vectors:* $\exists$ a connected component $\mathcal{T} \subset L^{-1}((0, \infty))$, with $\mathcal{T} \subset \mathcal{A}$, on which g^L has $(+, -, -, -)$ signature.

4) *Causal geodesics:* The Euler-Lagrange equations

$$\frac{d}{d\tau}\dot{\partial}_i L - \partial_i L = 0\,.$$

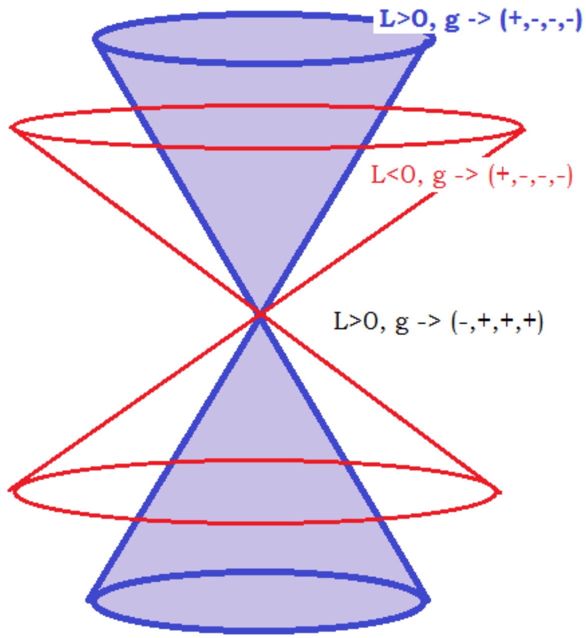
have a unique local solution for every initial condition $(x, \dot{x}) \in \mathcal{T} \cup \mathcal{N}$, where $\mathcal{N} = \ker L$.

Conic subbundles of TM:
- $\mathcal{A}$ - *admissible* vectors (g - exists and is smooth);
- $\mathcal{T}$ - future directed *timelike* vectors ($\mathcal{T} \subset \mathcal{A}$, $L > 0$, g is $(+,-,-,-)$);
- $\mathcal{N}$ - *null* vectors;

- $\mathcal{T} \subset \mathcal{A}$; $\partial\mathcal{T} \subset \mathcal{N}$; no relation required between $\mathcal{N}$ and $\mathcal{A}$.

- $F := \sqrt{|L|}$ - arc length element: $cd\tau = F(x, dx)$.

Examples of Finsler spacetimes: $L = F^2$,
- **Randers** type $F = \sqrt{|a_{ij}\dot{x}^i\dot{x}^j|} + b_i\dot{x}^i$, with $0 < g^{-1}(b,b) < 1$;
- **Bogoslovsky/Kropina** type $F = (|a_{ij}\dot{x}^i\dot{x}^j|)^{\frac{1-q}{2}}(b_k(x)\dot{x}^k)^q$, $g^{-1}(A,A) \geq 0$ - DSR
- **polynomial** m-th root type $F = |G_{a_1\cdots a_m}(x)\dot{x}^{a_1}\ldots\dot{x}^{a_m}|^{\frac{1}{m}}$ - birefringence.



light cone for a quartic Finsler metric

4 The stage for a Finsler gravity action

4.1 Finslerian fields: accommodating homogeneity

Consider an arbitrary fibered manifold $(\mathcal{F}, \pi_{\mathcal{F}}, \overset{\circ}{TM})$.

Finslerian field = a section $\Phi : \overset{\circ}{TM} \to \mathcal{F}, (x, \dot{x}) \mapsto \Phi(x, \dot{x})$, positively homogeneous of some degree k in $\dot{x}$.

Examples: a) Finsler function $L : \overset{\circ}{TM} \to \mathbb{R}$ $(k = 2)$;

b) 1-particle distribution function for a fluid $\varphi : \overset{\circ}{TM} \to \mathbb{R}$ $(k = 0)$.
Problem: **homogeneity** $\to$ we can't use classical variational principle on $\mathcal{F}$

Which configuration space to choose?

1) *Tangent bundle approach:* $X = \overset{\circ}{TM}$, $Y = \mathcal{F}$

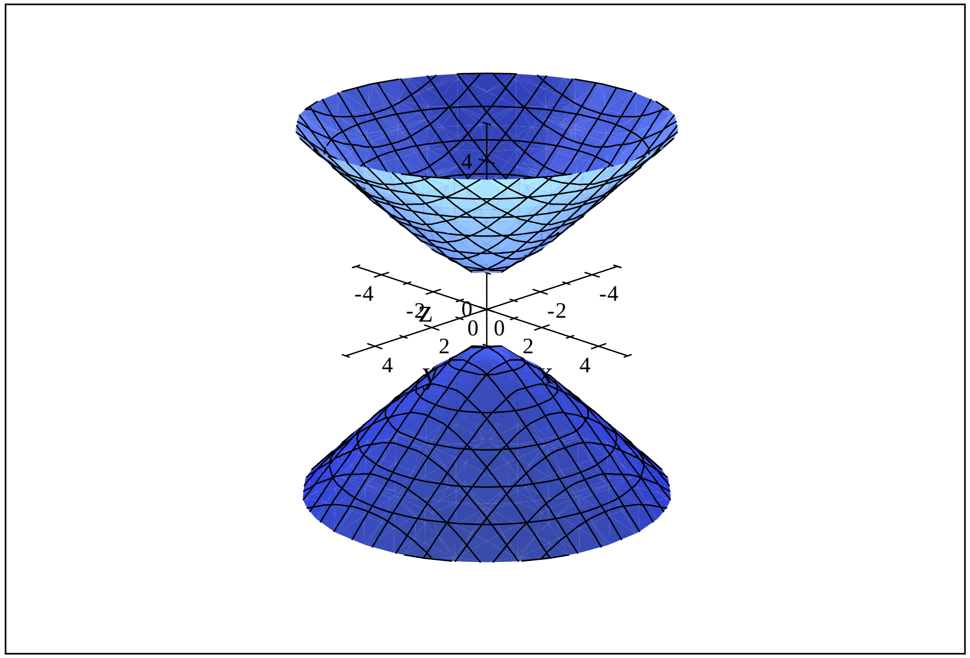
Problems: homogeneity $\rightarrow$ we can't set $\delta L(x, \dot{x}) = 0$ on the boundary ∂D;

- restrictions to be imposed on variations (s.th. the modified L is still 2-homogeneous).

2) *Unit sphere bundle (observer space) approach:* $X = SM$, $Y = \mathcal{F}$

$SM = \{(x, \dot{x}) \in TM \mid L(x, \dot{x}) = 1\}$.

Problems: $S_x M$ is non-compact (!) and depends on L:



Unit Lorentz-Finsler "sphere"

3) **Projectivised bundle approach** (to be discussed below):
- **no problem** in applying the classical variational principle
- *Finslerian fields can be naturally described as sections* of bundles sitting over PTM^+ (+ no restrictions needed on variations to preserve homogeneity of dynamical variable).

The positive projective bundle $PTM^+ := \{[(x,\dot{x})]_\sim \mid (x,\dot{x}) \in T\overset{\circ}{M}\}$:

$$(x,\dot{x}) \ \sim \ (x,u) \Leftrightarrow \ u = \lambda \dot{x} \ \text{ for some } \ \lambda > 0.$$

A point on $PTM^+ =$ a half-line in TM.

- PTM^+ is a compact, orientable 7-dimensional manifold.

- $(x^i, \dot{x}^i)$ - local *homogeneous coordinates* on PTM^+.

- $\mathbf{\overset{\circ}{TM}}$ **- principal bundle over** $\mathbf{PTM}^+$, with fiber: $(\mathbb{R}^*_+, \cdot)$:

$$\cdot \;: T\overset{\circ}{M} \times \mathbb{R}^*_+ \to T\overset{\circ}{M}, \qquad (x, \dot{x}) \cdot \alpha = (x, \alpha\dot{x})\,. \tag{1}$$

- $(T\overset{\circ}{M}, \pi^+, PTM^+)$ - principal bundle, with:

$$\pi^+ : T\overset{\circ}{M} \to PTM^+, \quad (x, \dot{x}) \mapsto [(x, \dot{x})]\,.$$

- **Calculus on** PTM^+ - performed identically to the one on TM

- Integrals on *compact subsets* $D^+ \subset PTM^+$ = integrals on *compact* subsets $D \subset SM$, where $D^+ = \pi^+(D)$.

M,C,N and the secret weapon: PTM^+

4.2 Finslerian fields as sections

Problem: $L = L(x, \dot{x})$, *not* $L([x, \dot{x}])$ *(!)*
How to describe it as a section $\gamma : PTM^+ \to Y$?

$(\mathbb{R}_+^*, \cdot)$ - Lie group, acting on both $\overset{\circ}{TM}$ and $\mathcal{F}$.
Remark. $\Phi : TM \to \mathcal{F}$ - Finslerian field $\Leftrightarrow$ Φ - *(equivariant map,)* w. r.t. the Lie group actions:

$$\begin{aligned} \cdot &: \overset{\circ}{TM \times} \mathbb{R}_+^* \to \overset{\circ}{TM}, \quad (x, \dot{x}) \cdot \alpha = (x, \alpha \dot{x}), \\ * &: \mathbb{R}_+^* \times \mathcal{F} \to \mathcal{F}, \alpha * z = \alpha^k z. \end{aligned}$$

Associated bundle (base - PTM^+, fiber - $\mathcal{F}$):

$$Y := (\overset{\circ}{TM} \times \mathcal{F})_{/\sim}$$

where $(x, \dot{x}, y^\sigma) \sim (x, \alpha\dot{x}, \alpha^k y^\sigma), \quad \alpha \in \mathbb{R}^*_+$.
Y - fibered manifold over PTM^+, with projection:

$$\pi : Y \to PTM^+, [(x^i, \dot{x}^i, y^\sigma] \to [(x, \dot{x})]$$

Local coords. on Y : $(x^i, u^\alpha, y^\sigma)$ - or $(x^i, \dot{x}^i, y^\sigma)$ (using homogeneous coords on PTM^+).

- **Finsler fields** $\Phi : TM \to \mathcal{F}$ - in one-to-one correspondence with **sections** of Y :

$$\Phi \mapsto \gamma : PTM^+ \to Y,\ \gamma([x, \dot{x}]) = [x, \dot{x}, \Phi(x, \dot{x})]. \qquad (2)$$

Finsler field Lagrangians:

$$\Lambda = \mathcal{L} Vol, \qquad \mathcal{L} = \mathcal{L}(x^i, \dot{x}^i, y^\sigma, ..., y^\sigma_{,i_1 ... i_l ... i_r}),$$

where $\Phi : (x^i, \dot{x}^i) \mapsto (y^\sigma(x^i, \dot{x}^i))$, $Vol = \mathbf{i}_{\mathbb{C}}(d^4x \wedge d^4\dot{x})$.

- Necessary condition: Λ - *0-homogeneous* 7-form.

5 A concrete vacuum action

Geodesics of (M, L): $$\frac{d^2x^i}{ds^2} + 2G^i{}_j(x, \frac{dx}{ds})\frac{dx^j}{ds} = 0. \qquad (*)$$

- $G^i{}_j(x, \dot{x})$ - coeffs. of **canonical non-linear connection**

$$N : T\mathcal{A} = H\mathcal{A} \oplus V\mathcal{A}.$$

- N induces a (nonlinear)covariant derivative: $\nabla : \Gamma(\mathcal{A}) \times \mathcal{X}(M) \to \mathcal{X}(M)$.

- Geodesic equation (*) then becomes: $\nabla_{\dot{c}}\dot{c} = 0.$

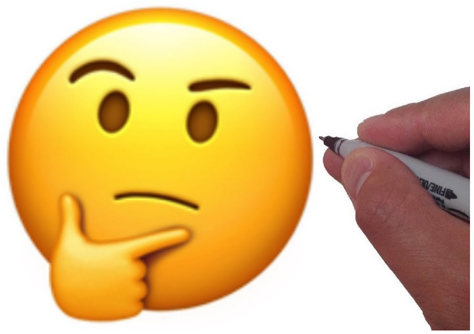
- Geodesic deviation equation: $\nabla_{\dot{c}}\nabla_{\dot{c}}\xi = \mathcal{R}(\dot{c}, \xi),$

- - $\mathcal{R}(\dot{c}, \xi) = R^i{}_j(\dot{c})\xi^j\partial_i$ - determined by first derivs of $G^i{}_j$.
  $R := R^i{}_i : \mathcal{A} \to \mathbb{R}$ - (non-homogenized) **Finslerian Ricci scalar**

Conjectured vacuum field equation (Rutz, 1993):

$$R = 0.$$

- Rutz eqn. is **not** variational $\Rightarrow$ which is its "closest" variational eqn?



$\mathcal{F} := \mathbb{R}$, $\quad k = 2$, $Y := (TM \times \mathbb{R}^*_+)_{/\sim}$

L - 2-homogeneous $\leftrightarrow$ section of Y : $\quad \gamma : (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, L(x, \dot{x}))$.

Start from Rutz equation $\longrightarrow$ find the "closest" Lagrangian

Technique: canonical variational completion (Voicu-Krupka, 2015)

- Start with arbitrary PDE system: $\varepsilon_\sigma(x^A, y^\mu, y^\mu{}_A, ...y^\mu{}_{A_1...A_r}) = 0$ (**)
- *Vainberg-Tonti Lagrangian density:*

$$\mathcal{L} = y^\sigma \int_0^1 \varepsilon_\sigma(x^A, ty^\mu, ty^\mu{}_A, ..., ty^\mu{}_{A_1...A_r})dt$$

VT-Lagrangian - built from ε_σ alone.

Properties of VT-Lagrangian:

1) If (**) - locally variational $\Rightarrow$ its Lagrangian density is, up to a total derivative term, $\mathcal{L}$.

2) Define *Helmholtz form* components:

$$H_\sigma := E_\sigma - \varepsilon_\sigma,$$

where

$$E_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_i(\frac{\partial \mathcal{L}}{\partial y^\sigma{}_i}) + ... + (-1)^r d_{i_1}...d_{i_r}(\frac{\partial \mathcal{L}}{\partial y^\sigma{}_{i_1...i_r}})$$

Helmholtz conditions: (**) is locally variational $\Longleftrightarrow$

$$H_\sigma = 0, \qquad \sigma = 1, ..., m. \tag{3}$$

Example: (M, g) - Lorentzian manifold. Variational completion of $r_{ij} = 0$ is:

$$r_{ij} - \frac{1}{2} r g_{ij} = 0.$$

Apply this technique to Rutz's equation $\Rightarrow$

$$\mathcal{L} = L^{-3} R \left|\det g\right| .$$

($\equiv$Pfeifer-Wohlfarth, 2011; Chen-Shen, 2008 - positive definite Finsler metrics)

Field equation:

$$3R - \frac{1}{2} g^{ij} R_{\cdot ij} + g^{ij}[(\nabla P_i)_{\cdot j} + P_{i|j} - P_i P_j] = 0.$$

Particular case: In Lorentzian spaces $(M, L = g_{ij}(x)\dot{x}^i\dot{x}^j)$, field eqn - equivalent to:

$$r_{ij} = 0.$$

6 Kinetic gas

- starting point: M. Hohmann, Int. J. Mod. Phys. A 31(2-3), 1641012 (2016)

- fundamental variable: **1-particle distribution function**

$$\varphi : \mathcal{O} \to \mathbb{R}, \varphi = \varphi(x, \dot{x})$$

- $\mathcal{O} = \{(x, \dot{x}) \in \mathcal{T} \mid L(x, \dot{x}) = 1\}$ - *observer space;*

-Number counting integral

$$N := \int_\sigma \varphi \Omega$$

gives the number of particle trajectories (C) which intersect a given hypersurface $\sigma \subset \mathcal{O}$

- Assumption: $\varphi_x = \varphi(x, \cdot)$ - compactly supported

- **Reeb vector field** $\mathbf{r} = l^i \delta_i$ - tangent to lifts

$$C : s \mapsto (x(s), \frac{dx}{ds}(s))$$

of *geodesics* of M.

Property: φ - obeys the *Liouville equation:* $\mathbf{r}(\varphi) = 0$.

Meaning of Liouville equation: φ is conserved along timelike geodesics of M.

7 Finsler description of kinetic gas

- Idea: Liouville equation = Finslerian *energy-momentum conservation*:

- Matter Lagrangian: $\Lambda_{matter} := \kappa\varphi\frac{|\det g|}{L^2}Vol_0 \Rightarrow$
- field equation:

$$3R - \frac{1}{2}g^{ij}R_{\cdot ij} + g^{ij}[(\nabla P_i)_{\cdot j} + P_{i|j} - P_i P_j] = \frac{\kappa}{2}\varphi$$

- **(!) No Riemannian solution on the support of** φ :

$$\begin{aligned} R_{\cdot ij} &= R_{\cdot ij}(x,\dot{x}) \quad inside \;\; supp(\varphi); \\ R_{\cdot ij} &= 0 \qquad\qquad\quad outside \; supp(\varphi). \end{aligned}$$

Define: **Energy-momentum source tensor** (on TM):

$$\mathbf{T}^i{}_j = \varphi u^i u_j$$

- $\mathbf{T}^i{}_j = \mathbf{T}^i{}_j(x, \dot{x})$ obey the *covariant conservation law*

$$\mathbf{T}^i{}_{j|i} = 0 \tag{4}$$

- Note (4) - is equivalent to the Liouville equation $\mathbf{r}(\varphi) = 0$;
- Usual energy-momentum tensor (on M) - obtained as:

$$T^i{}_j(x) := \int\limits_{\mathcal{O}_x} \mathbf{T}^i{}_j(x, \dot{x}) d\mathbf{\Sigma}_x = \int\limits_{\mathcal{O}_x} \varphi u^i u_j d\mathbf{\Sigma}_x$$

(consistent with: Zannias&Sarbach, 2014)

Conclusions:

1. Finsler field theories *are* mathematically well defined, once we choose as base manifold the (oriented) projective tangent bundle PTM^+.

2. We constructed a concrete vacuum action ($\longrightarrow$ Pfeifer-Wohlfarth equation)

3. Kinetic gas theory is naturally Finslerian - and offers a more accurate description of a fluid.

Future prospects:

- Construct and study a cosmological model; find observables.
- Define an averaging procedure $\longrightarrow$ could dark energy be (partially) an averaging effect?

Thank you !