



# Cosmographic reconstruction to discriminate between modified gravity and dark energy

**Salvatore Capozziello**

University of Naples "*Federico II*"

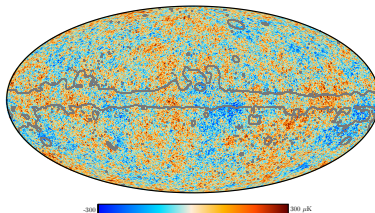
**CANTATA Meeting, Tuzla 2019**

- 1 The standard cosmological model:  $\Lambda$ CDM
- 2 Observational probes
- 3 Issues with the standard cosmological model
- 4 A model-independent approach: cosmography
  - Cosmography with Padé polynomials
  - Cosmography with Chebyshev polynomials
- 5 Unified Dark Energy Models
  - The case of Unified Anton-Schmidt Dark Energy
- 6 Modified Theories of Gravity
  - The case of  $f(T)$  teleparallel gravity
  - Reconstruction of the  $f(T)$  action
  - $f(R)$  gravity in the metric formalism
  - Reconstruction of the metric  $f(R)$  action
  - $f(R)$  gravity in the Palatini formalism
  - Reconstruction of the Palatini  $f(R)$  action
  - Comparison with  $\Lambda$ CDM
- 7 Conclusions and perspectives

# The standard cosmological model: $\Lambda$ CDM

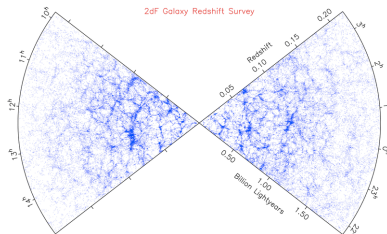
## Theoretical foundation: **Cosmological Principle**

- The Universe is spatially isotropic:



[Planck (2018)]

- The Universe is homogeneous on large scales:



- Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where  $a(t_0) = 1$  and

$$k = \begin{cases} -1, & \text{open universe} \\ 0, & \text{flat universe} \\ +1, & \text{closed universe} \end{cases}$$

- Energy-momentum tensor for a perfect fluid  $\nabla^\mu G_{\mu\nu} = 0 \implies \nabla^\mu T_{\mu\nu} = 0$  by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}.$$

- From  $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$ , Friedmann equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

with  $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$  and  $\dot{\rho} + 3H(\rho + p) = 0$

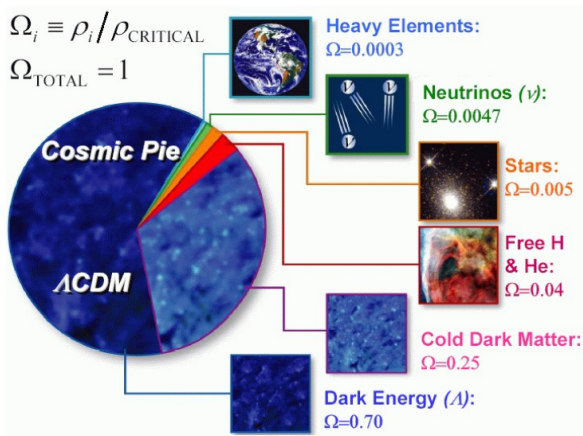
- Normalized density parameters:

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i, \quad \Omega_k = \frac{-k}{(aH)^2}, \quad \sum_i \Omega_i = 1.$$



## The concordance paradigm

- $\Lambda$ CDM model:  $\Omega_r \approx 0$  ,  $\Omega_k = 0$  ,  $\Omega_\Lambda = 1 - \Omega_m$



- **Supernovae Ia** (Pantheon sample):

$$\mu(z) = 25 + 5 \log_{10}[d_L(z)] , \quad d_L(z) = (1+z) \int_0^z \frac{dz'}{H(z')} .$$

The distance modulus is modelled as follows:

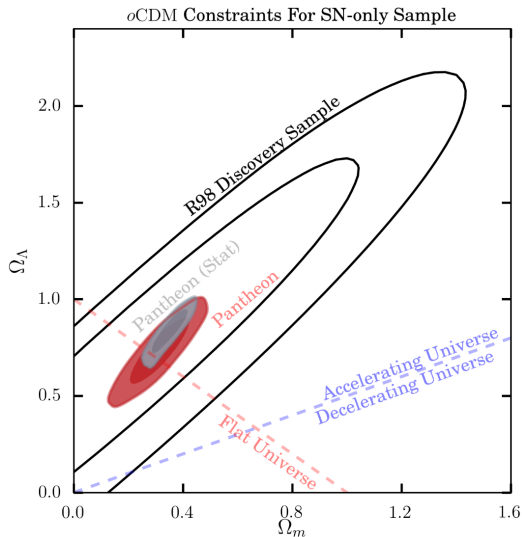
$$\mu_{obs} = m_B - M_B + \alpha x_1 - \beta c + \Delta_{bias} ,$$

- $m_B$  is the observed peak magnitude in the rest-frame  $B$  band;
- $x_1$  is the time stretching of the light-curve;
- $c$  is the supernova colour at maximum brightness;
- $\Delta_{bias}$  is a distance correction based on predicted biases from simulations;
- $M_B$  is the absolute magnitude defined on the host stellar mass:

$$M_B = \begin{cases} M, & \text{if } M_{host} < 10^{10} M_{Sun} \\ M + \Delta_M, & \text{otherwise} \end{cases}$$

- $\alpha$  and  $\beta$  are the fitting coefficients.

- The evidence for non-zero  $\Omega_\Lambda$  from the Pantheon SNeIa sample is  $> 6\sigma$  when including all systematic uncertainties.
- There is a factor of  $\sim 20$  improvement over the original constraints of Riess et al. (1998).
- The constraints are perfectly consistent with a flat universe.
- Without systematic uncertainties, the uncertainty on  $\Omega_m$  is roughly  $2\times$  smaller.



[Scolnic et al. (2018)]

- **CMBR power spectrum:**

Since the CMBR temperature anisotropy is a function over a sphere, it is possible to use a spherical harmonics expansion:

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) ,$$

where

$$Y_{\ell m}(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos \theta) e^{im\phi} ,$$

The variance  $\langle |a_{\ell m}|^2 \rangle$  represents the amplitude of the fluctuations of  $T$  for each multipole  $\ell$ :

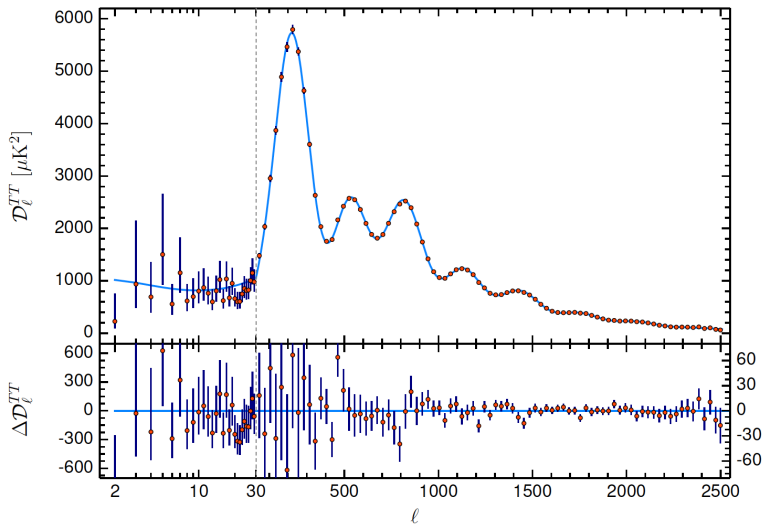
$$C_{\ell} \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell+1} \sum_m a_{\ell m}^* a_{\ell m}$$

For Gaussian perturbations, the  $C_{\ell}$ s contain all the statistical information about the CMBR temperature anisotropy:

- The term  $\ell = 0$  is the monopole and corresponds to the average temperature;
- The dipole,  $\ell = 1$ , is due to the motion of the Solar System relative to the CMBR rest frame and it is then a Doppler effect;
- The  $\ell \geq 2$  multipoles carry information about the intrinsic anisotropies of the CMBR.

# Planck 2018 temperature power spectrum

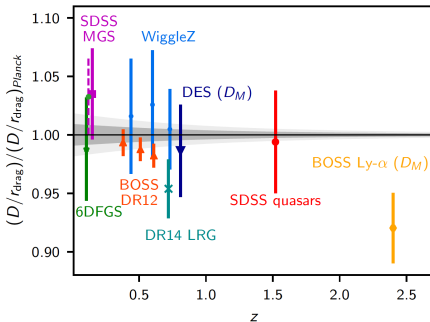
- Amplitude of the CMBR power spectrum:  $\mathcal{D}_\ell \equiv \ell(\ell + 1)C_\ell/2\pi$ .
- The  $\Lambda$ CDM model is in remarkable agreement with the Planck likelihood.



- **BAO measurements:**

The large-scale structure of the Universe has been extensively studied via redshift surveys. The estimated galaxy correlation function shows a peak at large scales which is interpreted as the signature of the baryon acoustic oscillation in the relativistic plasma of the early Universe. It is standard to quantify this phenomenon by estimating the combination of the comoving sound horizon at the drag epoch  $r_{\text{drag}}$ , and the spherically averaged distance  $D_V$ :

$$D_V(z) = \left[ D_M^2(z) \frac{z}{H(z)} \right]^{1/3}, \quad D_M = \frac{d_L}{1+z}$$



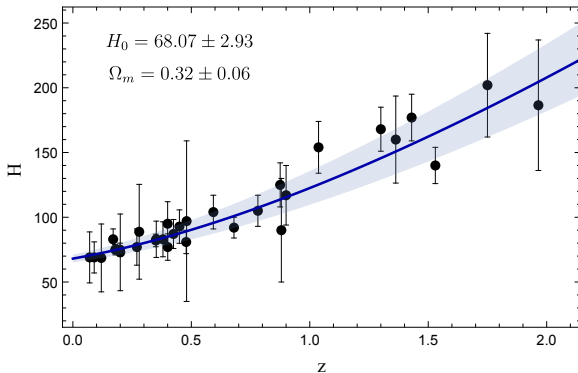
The BAO acoustic scale ( $\sim 147$  Mpc) is much larger than the scale of virialized structures. Due to this separation of scales, BAO measurements are insensitive to nonlinear physics and provide a robust geometrical test of cosmology.

[Planck (2018)]

- **Observational Hubble data:**

The Hubble rate of a given cosmological model can be constrained by means of the model-independent measurements acquired through the differential age method [Jimenez & Loeb (2002)]. Such a technique uses red passively evolving galaxies as cosmic chronometers. Once the age difference of galaxies at two close redshifts is measured, one can obtain  $H(z)$  from the relation

$$\frac{dt}{dz} = -\frac{1}{(1+z)H(z)} .$$



- **Growth rate:**

In the context of linear metric perturbations to the FLRW background space-time, one can study the evolution of matter density fluctuations on sub-horizon scales:

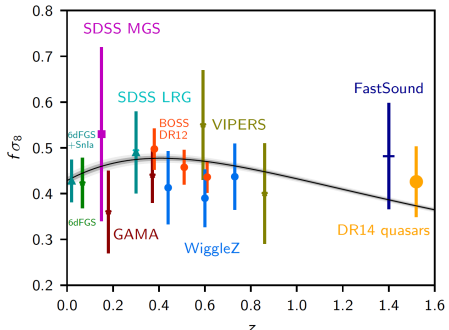
$$\delta_m''(a) + \left[ \frac{3}{a} + \frac{H'(a)}{H(a)} \right] \delta_m'(a) - \frac{3}{2} \frac{\Omega_m H_0^2}{a^5 H^2(a)} \delta_m(a) = 0 ,$$

where  $\delta_m \equiv \delta\rho_m/\rho_m$  is the density contrast. The growth rate of the density perturbations is described by the quantity  $f(a) = d\delta_m/d\ln a$ . Measurements from redshift-space distortion and weak lensing have been obtained for the factor

$$f\sigma_8(z) \equiv f(z)\sigma_8(z) ,$$

where  $\sigma_8(a) = \sigma_8\delta_m(a)/\delta_m(1)$  is the rms fluctuations of the linear density field inside a radius of  $8h^{-1}\text{Mpc}$ , and  $\sigma_8$  is its present day value.

The plot shows the Planck  $\Lambda\text{CDM}$  cosmology fitted to the growth rate of fluctuations from various redshift surveys. These results provide the tightest constraints to date on the growth rate of fluctuations.



[Planck (2018)]



# The cosmological constant problem

## Fine-tuning

- Energy scales (units of  $c = \hbar = k_B = 1$ ):

$$M_{Pl} = G^{-1/2} \approx 10^{19} \text{ GeV} , \quad H_0 \approx 10^{-42} \text{ GeV}$$

- FLRW cosmology:

$$\rho_\Lambda = \Lambda M_{Pl}^2 \simeq H_0^2 M_{Pl}^4 \approx 10^{-46} \text{ GeV}^4$$

- Quantum field theory:

$$\rho_{vac} \sim M_{Pl}^4 \approx 10^{76} \text{ GeV}^4$$

$$\rho_{vac} \sim 10^{122} \rho_\Lambda$$

## Coincidence

- Very different evolution histories:

$$\frac{\Omega_\Lambda}{\Omega_m} = \frac{\rho_\Lambda}{\rho_m} \propto a^3$$

- But observations indicate:

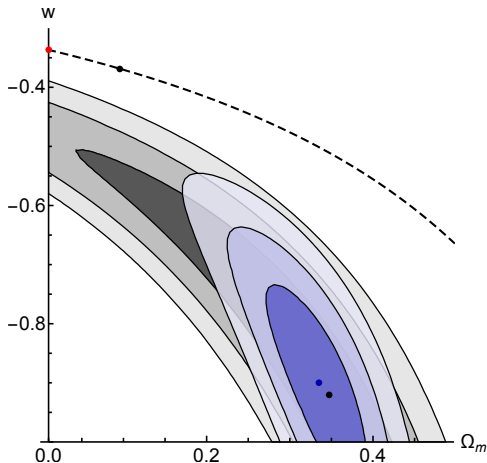
$$\Omega_\Lambda \simeq 0.7 , \quad \Omega_m \simeq 0.3$$

## A prototype of extensions: the $w$ CDM model

Relax the assumption of cosmological constant ( $w = -1$ ) and allow the dark energy EoS parameter to vary  $\Rightarrow w$ CDM model:

$$H(z) = H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}}$$

- Combining CMBR and SNela:  
 $w = -1.026 \pm 0.041$ . This is consistent with the cosmological constant model.
- Combining CMBR and BAO:  
 $w = -0.991 \pm 0.074$ . This is consistent with CMBR+SNela, though less precise.
- Combining CMBR, SNela, BAO and local  $H_0$  measurements:  
 $w = -1.047 \pm 0.038$ . This is similar to just CMBR+SNela but consistent with the cosmological constant only at the  $2\sigma$  level.



Consider a time-evolving equation of state for dark energy:

$$H(a) = H_0 \left[ \frac{\Omega_m}{a^3} + \Omega_{DE} \exp \left\{ -3 \int_1^a [1 + w_{DE}(a')] d \ln a' \right\} \right]^{1/2}.$$

A simple parameterization of  $w(a)$  is obtained by a first-order Taylor expansion:

$$w_{DE} = w_0 + w_a(1 - a).$$

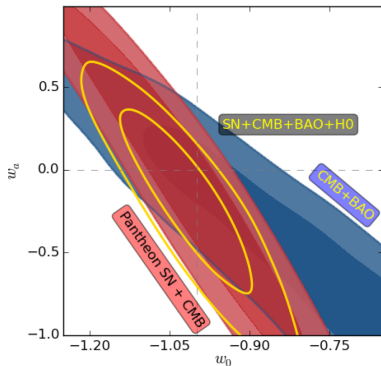
This is the Chevallier-Polarski-Linder (CPL) model, which well-behaves from very high redshift ( $w(1) = w_0 + w_a$ ) to the present epoch ( $w(0) = w_0$ ). Such a parameterization is capable of reproducing with high accuracy the EoS of several scalar field models, as well as the resulting distance-redshift relations.

Combining CMBR, SNela, BAO and local  $H_0$  measurements:

$$w_0 = -1.007 \pm 0.089$$

$$w_a = -0.222 \pm 0.407$$

These results are consistent with the cosmological constant model ( $w_0 = -1$ ,  $w_a = 0$ ), indicating no evidence for evolution of the dark energy equation of state.



[Scolnic et al. (2018)]

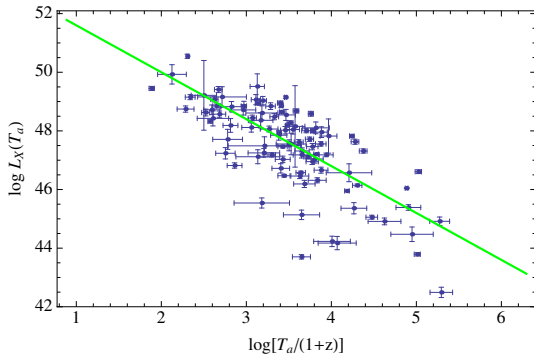
## Constraints on the $w$ CDM model

Sample	$w$	$\Omega_m$	$H_0$
CMB+BAO	$-0.991 \pm 0.074$	$0.312 \pm 0.013$	$67.508 \pm 1.633$
CMB+H0	$-1.188 \pm 0.062$	$0.265 \pm 0.013$	$73.332 \pm 1.729$
CMB+BAO+H0	$-1.119 \pm 0.068$	$0.289 \pm 0.011$	$70.539 \pm 1.425$
SN+CMB	$-1.026 \pm 0.041$	$0.307 \pm 0.012$	$68.183 \pm 1.114$
SN+CMB+BAO	$-1.014 \pm 0.040$	$0.307 \pm 0.008$	$68.027 \pm 0.859$
SN+CMB+H0	$-1.056 \pm 0.038$	$0.293 \pm 0.010$	$69.618 \pm 0.969$
SN+CMB+BAO+H0	$-1.047 \pm 0.038$	$0.299 \pm 0.007$	$69.013 \pm 0.791$

## Constraints on the CPL model

Sample	$w_0$	$w_a$	$\Omega_m$	$H_0$
CMB+BAO	$-0.616 \pm 0.262$	$-1.108 \pm 0.771$	$0.343 \pm 0.025$	$64.614 \pm 2.447$
CMB+H0	$-1.024 \pm 0.347$	$-0.789 \pm 1.338$	$0.265 \pm 0.015$	$73.397 \pm 1.961$
CMB+BAO+H0	$-0.619 \pm 0.270$	$-1.098 \pm 0.781$	$0.343 \pm 0.026$	$64.666 \pm 2.526$
SN+CMB	$-1.009 \pm 0.159$	$-0.129 \pm 0.755$	$0.308 \pm 0.018$	$68.188 \pm 1.768$
SN+CMB+BAO	$-0.993 \pm 0.087$	$-0.126 \pm 0.384$	$0.308 \pm 0.008$	$68.076 \pm 0.858$
SN+CMB+H0	$-0.905 \pm 0.101$	$-0.742 \pm 0.465$	$0.287 \pm 0.011$	$70.393 \pm 1.079$
SN+CMB+BAO+H0	$-1.007 \pm 0.089$	$-0.222 \pm 0.407$	$0.300 \pm 0.008$	$69.057 \pm 0.796$

[Scolnic et al. (2018)]



**Figure:** Best fit curves for the  $L_X^* - T_a^*$  correlation relation superimposed on the data in our full bayesian approach, as it results for the FlatCPL model and GRB + SNela +  $H(z)$  +  $H_0$  datasets. See Dainotti, Cardone, Capozziello MNRAS Lett. 2008.

[Dainotti et al. MNRAS (2013)]

[Cardone et al. MNRAS (2009,2010)]

[Postnikov et al. ApJ (2014)]

# But is that all? Further issues with the standard cosmological model!

## Dark matter

- New particles seem to be elusive in laboratories.
- Cosmology points out for new physics.
- No WIMPs?

## $H_0$ tension

- Riess et al. measured the normalized Hubble rate:  $h_0 \sim 0.73$ .
- Planck measurements are smaller:  $h_0 \sim 0.68$ .
- Errors cannot account for a matching between the two measurements.

# The $H_0$ tension

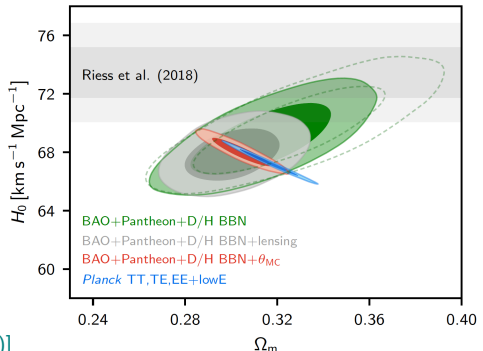
Another problem that compromises our understanding of the cosmic speed up concerns the discrepancy between the model-dependent and the direct measurements of the present expansion rate of the universe. Using the period-luminosity relation for Cepheids to calibrate a number of secondary distance indicators such as SNIa, Riess et al. (2018) estimate:

$$H_0 = (73.48 \pm 1.66) \text{ km s}^{-1} \text{ Mpc}^{-1}.$$

This value is in  $3.5\sigma$  tension with that of the CMBR-Planck 2018  $\Lambda$ CDM model:

$$H_0 = (67.27 \pm 0.60) \text{ km s}^{-1} \text{ Mpc}^{-1}.$$

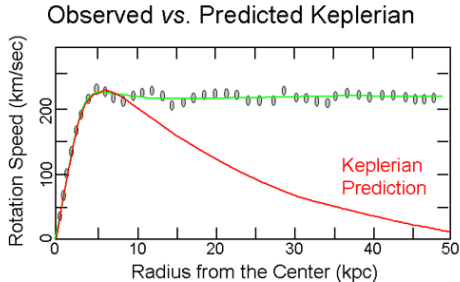
- The tension is not confined exclusively to the Planck results.
- The constraints on  $H_0$  and  $\Omega_m$  converge to the Planck values as more data are included.
- If the difference between Planck and the R18 measurements of  $H_0$  is caused by new physics, then it is unlikely to be through some change to the late-time distance-redshift relation.



[Planck (2018)]

# The dark matter problem

Galaxies are rotating with such a speed that gravity generated by their observed matter could not hold them together; they should have torn themselves apart long ago.



**The microphysics behind dark matter → corrections to Einstein's equations with no new particles?**

- Dark matter is essential to account for structure formation.
- Its nature is cold and it does not provide indications for standard particles at LHC.
- Rotational curves in spirals are dramatically flat.



## Barotropic unified models of dark energy and dark matter

- A first prototype: Chaplygin gas and its extensions:

$$P = -\frac{A}{\rho}, \quad P = B\rho^\gamma - \frac{A}{\rho^\alpha}.$$

- Logotropic dark energy models

$$P = -\sigma \log \rho.$$

## Alternatives to general relativity

- Extensions of Einstein's gravity

$$R \rightarrow f(R), \quad R \rightarrow f(R, G), \quad R \rightarrow f(R, \square R), \quad R \rightarrow \text{Scalar-Tensor}.$$

- Modified gravity

$$R \rightarrow T, \quad T \rightarrow f(T).$$

**Which is the one which solves the issues of the concordance paradigm? → Cosmography as a tool to discriminate among models!!!**

# A model-independent approach: cosmography

- Taylor expansion of the scale factor (assuming flat FLRW universe):

$$a(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left. \frac{d^k a}{dt^k} \right|_{t=t_0} (t - t_0)^k$$

- Cosmographic series:

$$\begin{aligned} H(t) &\equiv \frac{1}{a} \frac{da}{dt} , & q(t) &\equiv -\frac{1}{aH^2} \frac{d^2 a}{dt^2} \\ j(t) &\equiv \frac{1}{aH^3} \frac{d^3 a}{dt^3} , & s(t) &\equiv \frac{1}{aH^4} \frac{d^4 a}{dt^4} \end{aligned}$$

- Luminosity distance:

$$d_L(z) = (1+z) \int_0^z \frac{dz'}{H} = \frac{1}{H_0} (c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4) + \mathcal{O}(z^5)$$

- Hubble expansion rate:

$$H(z) = \left[ \frac{d}{dz} \left( \frac{d_L(z)}{1+z} \right) \right]^{-1} = H_0 \left[ 1 + H^{(1)} z + H^{(2)} \frac{z^2}{2} + H^{(3)} \frac{z^3}{6} \right] + \mathcal{O}(z^4)$$

$$H^{(1)} = 1 + q_0 , \quad H^{(2)} = j_0 - q_0^2 , \quad H^{(3)} = 3q_0^2 + 3q_0^3 - j_0(3 + 4q_0) - s_0$$

[Cattoen, Visser, PRD, 78, 063501 (2008)]

[Capozziello, Lazkoz, Salzano, PRD, 84, 124061 (2011)]

- Limits of standard cosmography:

- the radius of convergence of the Taylor series is restricted to  $|z| < 1$ ;
- if cosmological data for  $z > 1$  are used, the Taylor series does not provide a good approximation of the luminosity distance due to its divergent behavior;
- finite truncations cause errors propagation that may result in possible misleading outcomes.

- Advantages of rational polynomials:

- they extend the radius of convergence of Taylor series;
- they can better approximate situations at high-redshift domains;
- the series can be modelled by choosing appropriate orders depending on each case of interest.

# Cosmography with Padé polynomials

- Series expansion of a generic function:  $f(z) = \sum_{k=0}^{\infty} c_k z^k, c_k = f^{(k)}(0)/k!$
- $(N, M)$  Padé polynomial:

$$P_{N,M}(z) = \frac{\sum_{n=0}^N a_n z^n}{1 + \sum_{m=1}^M b_m z^m}, \quad \begin{cases} P_{N,M}(0) = f(0) \\ P'_{N,M}(0) = f'(0) \\ \vdots \\ P_{N,M}^{(N+M)}(0) = f^{(N+M)}(0) \end{cases}$$

- $N + M + 1$  unknown coefficients:

$$\sum_{k=0}^{\infty} c_k z^k = \frac{\sum_{n=0}^N a_n z^n}{1 + \sum_{m=1}^M b_m z^m} + \mathcal{O}(z^{N+M+1})$$

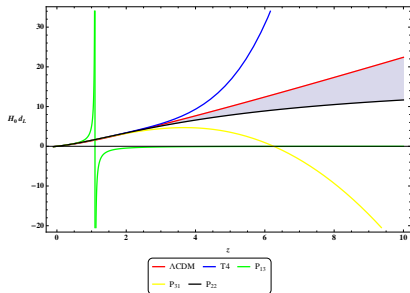
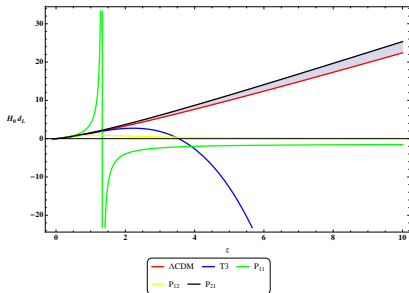
$$(1 + b_1 z + \dots + b_M z^M)(c_0 + c_1 z + \dots) = a_0 + a_1 z + \dots + a_N z^N + \mathcal{O}(z^{N+M+1})$$

- $(N, M)$  Padé approximation of the luminosity distance:

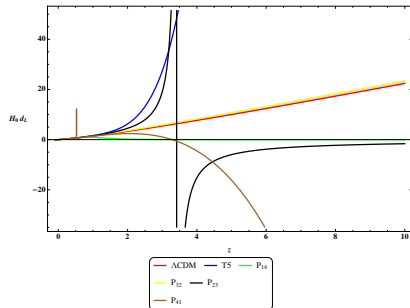
$$d_L(z) \approx P_{N,M}(z, H_0, q_0, j_0, s_0, \dots)$$

[Capozziello, Ruchika, Sen, MNRAS, 484, 4484 (2019)]

# Taylor vs Padé



- The Taylor polynomials  $T_3$ ,  $T_4$  and  $T_5$  rapidly diverge from the exact  $\Lambda$ CDM curve as  $z > 2$ .
- Padé polynomials  $P_{11}$ ,  $P_{13}$  and  $P_{23}$  give spurious singularities when used to approximate the  $\Lambda$ CDM model.
- The Padé functions  $P_{21}$ ,  $P_{22}$  and  $P_{32}$  fairly approximate the exact  $\Lambda$ CDM luminosity distance over the whole interval considered.



[Aviles, et al. PRD, 87, 064025 (2014)]

- Chebyshev polynomials of the first kind:

$$T_n(z) = \cos(n\theta) , \quad n \in \mathbb{N}_0 , \quad \theta = \arccos(z)$$

- They form an orthogonal set with respect to the weighting function  $w(z) = (1 - z^2)^{-1/2}$  in the domain  $|z| \leq 1$ :

$$\int_{-1}^1 T_n(z) T_m(z) w(z) dz = \begin{cases} \pi , & n = m = 0 \\ \frac{\pi}{2} \delta_{nm} , & \text{otherwise} \end{cases}$$

- Recurrence relation:

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$$

- Chebyshev series of a generic function  $f(z)$ :

$$f(z) = \sum_{k=0}^{\infty} ' c_k T_k(z)$$

where  $\sum'$  means that the first term in the sum must be divided by 2, and

$$c_k = \frac{2}{\pi} \int_{-1}^1 g(z) T_k(z) w(z) dz$$

being  $g(z)$  the Taylor series of  $f(z)$  around  $z = 0$ .

- Construct the  $(n, m)$  rational Chebyshev approximation of  $f(z)$ :

$$R_{n,m}(z) = \frac{\sum_{i=0}^n a_i T_i(z)}{\sum_{j=0}^m b_j T_j(z)}$$

- Requiring  $f(z) - R_{n,m}(z) = \mathcal{O}(T_{n+m+1})$ :

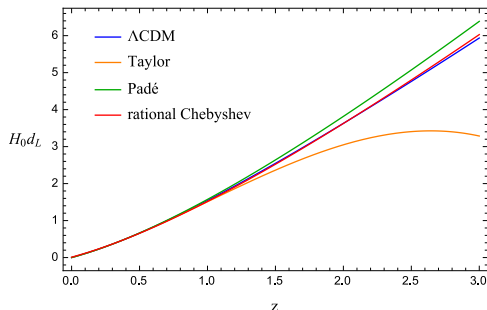
$$\begin{cases} a_i = \frac{1}{2} \sum_{j=0}^m b_j (c_{i+j} + c_{|i-j|}) = 0, & i = 0, \dots, n \\ \sum_{j=0}^m b_j (c_{i+j} + c_{|i-j|}) = 0, & i = n+1, \dots, n+m \end{cases}$$

- Generalization to  $z \in [a, b]$ :  $z = \frac{a(1 - \cos \theta) + b(1 + \cos \theta)}{2}$

$$T_n^{[a,b]}(z) = T_n \left( \frac{2z - (a+b)}{b-a} \right)$$

which are orthogonal with respect to  $w_{[a,b]} = [(z-b)(b-z)]^{-1/2}$ .

[Capozziello, D'Agostino, Luongo, MNRAS, 476, 3924 (2018)]



## Comparison among different cosmographic techniques

**Figure:** (2,1) rational Chebyshev approximation of the luminosity distance for the  $\Lambda$ CDM model with the correspondent Padé and Taylor approximations.

Parameter	Taylor			Padé			Rational Chebyshev		
	Mean	$1\sigma$	R.E.	Mean	$1\sigma$	R.E.	Mean	$1\sigma$	R.E.
$H_0$	65.80	$^{+2.09}_{-2.11}$	3.19%	64.94	$^{+2.11}_{-2.02}$	3.17%	64.95	$^{+1.89}_{-1.94}$	2.95%
$q_0$	-0.276	$^{+0.043}_{-0.049}$	16.8%	-0.285	$^{+0.040}_{-0.046}$	15.1%	-0.278	$^{+0.021}_{-0.021}$	7.66%
$j_0$	-0.023	$^{+0.317}_{-0.397}$	1534%	0.545	$^{+0.463}_{-0.652}$	102%	1.585	$^{+0.497}_{-0.914}$	44.5%

**Table:** 68% confidence level constraints and relative errors from the MCMC analysis of SN+OHD+BAO data for the fourth-order Taylor, (2,2) Padé and (2,1) rational Chebyshev polynomial approximations of the luminosity distance.

[Capozziello, D'Agostino, Luongo, MNRAS, 476, 3924 (2018)]



**The above approaches are model independent because based only on the convergence of polynomials. They can be used to discriminate among Dark Energy and Modified Gravity scenarios. In other words, cosmography can be used as a tool to probe new physics.**

- The idea is to combine Dark Matter and Dark Energy behaviors under the same standard.
- Dark Matter means the clustering properties of large scale structure.
- Dark Energy means reproducing the accelerated behavior of the Hubble flow.
- The goal is to reconstruct the cosmic history matching decelerated (matter dominance) and accelerated (dark energy dominance) behaviors at any redshift.
- Using cosmography at late ( $z \simeq 0$ ) and early ( $z \gg 0$ ) epochs.

# The case of Unified Anton-Schmidt Dark Energy

- Crystalline solid's pressure under isotropic deformation in the Debye approximation:

$$p(V) = -\beta \left( \frac{V}{V_0} \right)^{-\frac{1}{6} - \gamma_G} \ln \left( \frac{V}{V_0} \right)$$

$$\gamma_G < -\frac{1}{6} : \begin{cases} V < V_0, & \text{vanishing pressure, matter-dominated phase} \\ V = V_0, & \text{transition epoch} \\ V > V_0, & \text{negative pressure, accelerated phase.} \end{cases}$$

- Assume the universe filled with a **single fluid** obeying the Anton-Schmidt EoS ( $V \propto \rho^{-1}$ ):

$$p(\rho) = A \left( \frac{\rho}{\rho_*} \right)^{-n} \ln \left( \frac{\rho}{\rho_*} \right)$$

- Integrating the first law of thermodynamics for an adiabatic fluid:

$$\epsilon = \rho - \left[ \frac{A}{n+1} \left( \frac{\rho}{\rho_*} \right)^{-n} \ln \left( \frac{\rho}{\rho_*} \right) + \frac{A}{(n+1)^2} \left( \frac{\rho}{\rho_*} \right)^{-n} \right] \equiv \epsilon_m + \epsilon_{de}$$

- $\epsilon_m$  is the rest-mass energy, mimics (baryonic + dark) matter;
- $\epsilon_{de}$  is the internal energy, mimics dark energy.

- Early times ( $\rho \gg 1$ ):  $\epsilon_m$  dominates and, for  $n < 0$ ,  $p \ll \epsilon$
- Late times ( $\rho \ll 1$ ):  $\epsilon_{de}$  dominates and, for  $n < 0$ ,  $p \rightarrow -K$  ( $K > 0$ )

[Capozziello, D'Agostino, Luongo, PDU, 20, 1 (2018)]

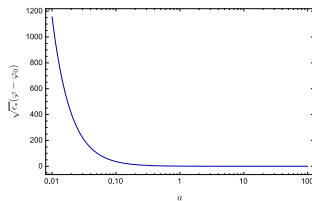
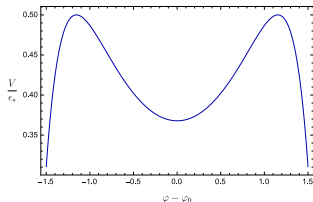
# The case of Unified Anton-Schmidt Dark Energy

Since  $\varphi = \varphi_0 + \frac{2}{\xi\sqrt{3\epsilon_*}} \left(\frac{C}{a^3}\right)^{\frac{\xi}{2}}$ , the potential:

$$V(\varphi) = \epsilon_* \left[ 1 - \frac{3}{8}\xi^2(\varphi - \varphi_0)^2 \right] \exp \left\{ -\frac{1}{\xi} + \frac{3}{4}\xi(\varphi - \varphi_0)^2 \right\}.$$

Symmetry around  $\varphi_0$ . It takes the values:

$$V_{min} = \epsilon_* e^{-1/\xi} \quad V_{max} = \frac{1}{2}\epsilon_* \xi e^{-1+1/\xi}.$$



The equation of state  $\omega_\varphi = -1 + \frac{3}{4}\xi^2(\varphi - \varphi_0)^2$ . As  $\xi \ll 1 \rightarrow \omega_\varphi \rightarrow -1$ .

[Capozziello, Giambó, D'Agostino, Luongo, PRD, 99, 023532 (2019)]

- Instead of searching for new particles, extend or modify GR.
- Dark Energy and Dark Matter as geometric effects at infrared scales.
- Extended Gravity means that GR is reproduced in a given regime, e.g.  $f(R) \rightarrow R$ .
- Modified Gravity means that standard GR could not be reproduced.
- Teleparallel Equivalent General Relativity (TEGR), gravitational field is represented by torsion  $T$  instead of curvature  $R$ , e.g.  $f(T) \rightarrow T$ .
- Cosmography could discriminate for new physics.

# The case of $f(T)$ teleparallel gravity

- Vierbein fields:  $e_A(x^\mu)$
- Metric of tangent space:  $\eta_{AB} = \text{diag}(+1, -1, -1, -1)$
- Space-time metric:  $g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu$
- **Zero-curvature** Weitzenböck connections:  $\hat{\Gamma}^\lambda_{\mu\nu} = e^\lambda_A \partial_\mu e^A_\nu$
- **Torsion** tensor:  $T^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\nu\mu} = e^\lambda_A (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu)$
- Action:

$$S = \int d^4x \, e \left[ \frac{f(T)}{2} + \mathcal{L}_m \right], \quad e = \sqrt{-g} = \det(e^A_\mu)$$

- Field equations:

$$e^\rho_A S^\mu{}_\rho (\partial_\mu T) f'' + \left[ \frac{1}{e} \partial_\mu (e e^\rho_A S^\mu{}_\rho) - e^\lambda_A T^\rho{}_{\mu\lambda} S^\mu{}_\rho \right] f' + \frac{1}{4} e^\nu_A f = \frac{1}{2} e^\rho_A T^{(m)}{}_\rho{}^\nu$$

- Flat FLRW metric  $\Longleftrightarrow e^\mu_A = \text{diag}(1, a, a, a)$
- Modified Friedmann equations:

$$H^2 = \frac{1}{3}(\rho_m + \rho_T)$$

$$2\dot{H} + 3H^2 = -\frac{1}{3}(p_m + p_T)$$

- Matter contribution:  $\rho_m \propto a^{-3} = (1+z)^3$  ,  $p_m = 0$
- Torsion contribution:

$$\rho_T = T f'(T) - \frac{f(T)}{2} - \frac{T}{2} , \quad p_T = \frac{f - T f'(T) + 2T^2 f''(T)}{2[f'(T) + 2T f''(T)]}$$

- Effective dark energy given by torsion where EoS is:

$$w_{de} \equiv \frac{p_T}{\rho_T} = -1 + \frac{(f - 2T f')(f' + 2T f'' - 1)}{(f + T - 2T f')(f' + 2T f'')}$$

- Friedmann equations:

$$H^2 = -\frac{1}{12f'(T)} [T\Omega_{m0}(1+z)^3 + f(T)]$$

$$\dot{H} = \frac{1}{4f'(T)} \left[ T\Omega_{m0}(1+z)^3 - 4HT\dot{f}''(T) \right]$$

- Relation between torsion scalar and Hubble parameter:

$$T = -6H^2$$

[Capozziello, Cardone, Farajollahi, Ravanpak, PRD, 84, 043527 (2011)]

[D'Agostino, Luongo, PRD, 98, 124013 (2018)]

[Abedi, Capozziello, D'Agostino, Luongo, PRD, 97, 084008 (2018)]

# Reconstruction of the $f(T)$ action

- Combine the Friedmann equations:

$$\left(\frac{df}{dz}\right)^{-1} \left[ H(1+z) \frac{d^2 f}{dz^2} + 3f \frac{dH}{dz} \right] = \frac{1}{H} \left(\frac{dH}{dz}\right)^{-1} \left[ 3 \frac{dH}{dz} + (1+z) \frac{d^2 H}{dz^2} \right]$$

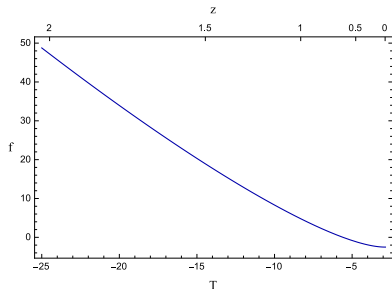
- Initial conditions:

1.  $G_{\text{eff}} \equiv G_N / f'(T) = G_N \implies \left. \frac{df}{dz} \right|_{z=0} = 1$

2. First Friedmann equation at  $z = 0$  &  $T = -6H^2$ :

$$f(T(z=0)) = f(z=0) = 6H_0^2(\Omega_{m0} - 2)$$

- Invert  $H(z)$  and use  $T = -6H^2$  to find  $z(T)$
- Plug  $z(T)$  into  $f(z)$  to find  $f(T)$



**Figure:** Numerical reconstruction of the  $f(T)$  function obtained from the third-order Taylor approximation of the luminosity distance.

[Capozziello, D'Agostino, Luongo, GeRG, 49, 141 (2017)]



## $f(R)$ gravity in the metric formalism

- Action:

$$S = \int d^4x \sqrt{-g} \left[ \frac{f(R)}{2} + \mathcal{L}_m \right]$$

- Varying the action with respect to  $g_{\mu\nu}$ :

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(curv)}$$

- Matter energy-momentum tensor:

$$T_{\mu\nu}^{(m)} = \frac{-2}{f' \sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad f' \equiv \frac{df}{dR}$$

- Effective curvature energy-momentum tensor:

$$T_{\mu\nu}^{(curv)} = \frac{1}{f'} \left[ \frac{1}{2} g_{\mu\nu} (f - R f') + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f' \right]$$

- Flat FLRW metric:

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j$$

- Relation between the Ricci scalar and the Hubble parameter:

$$R = -6(\dot{H} + 2H^2)$$

- Matter energy-momentum tensor for a perfect fluid:

$$T_{\mu\nu}^{(m)} = \text{diag}(\rho, -p, -p, -p)$$

- Modified Friedmann equations:

$$H^2 = \frac{1}{3} \left[ \frac{1}{f'} \rho_m + \rho_{\text{curv}} \right]$$

$$2\dot{H} + 3H^2 = -\frac{p_m}{f'} - p_{\text{curv}}$$

where

$$\rho_{\text{curv}} = \frac{1}{f'} \left[ \frac{1}{2} (f - Rf') - 3H\dot{R}f'' \right]$$

$$p_{\text{curv}} = \frac{1}{f'} \left[ 2H\dot{R}f'' + \ddot{R}f'' + \dot{R}^2 f''' - \frac{1}{2} (f - Rf') \right]$$

- Effective dark energy given by curvature. EoS is:

$$w_{de} \equiv \frac{p_{\text{curv}}}{\rho_{\text{curv}}} = -1 + \frac{\ddot{R}f'' + \dot{R}^2 f''' - H\dot{R}f''}{(f - Rf')/2 - 3H\dot{R}f''}$$

- Assuming matter as dust:

$$p_m = 0, \quad \rho_m = \frac{\rho_{m0}}{a^3} = 3H_0^2 \Omega_{m0} (1+z)^3$$

# Reconstruction of the metric $f(R)$ action

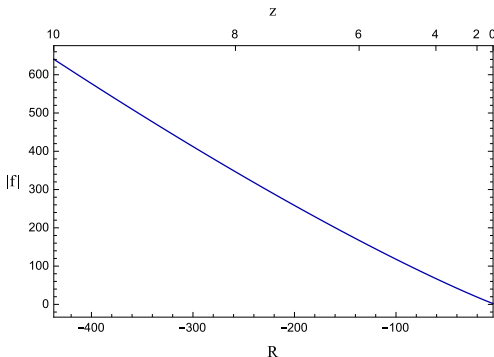
- Combining first Friedmann equation and  $R = -6(\dot{H} + 2H^2)$ :

$$\mathfrak{F}(z, f, f_z, f_{zz}) = 0$$

- Assuming  $f'(R_0) = 1$  ( $G_{\text{eff}} = G_N/f'(R)$ ), the initial conditions are:

$$f_0 = R_0 + 6H_0^2(\Omega_{m0} - 1), \quad f_z|_{z=0} = R_z|_{z=0}$$

- Use  $R = -6(\dot{H} + 2H^2)$  with  $H_{2,1}(z)$  to get  $R(z)$
- Invert  $R(z)$  and plug into  $f(z)$  to obtain  $f(R)$



**Figure:** Numerical reconstruction of  $f(R)$  using the (2,1) Padé approximation of the Hubble rate.

[Capozziello, D'Agostino, Luongo, JCAP, 1805, 008 (2018)]

## Viability conditions for $f(R)$ models

1. Avoid negative values of  $G_{\text{eff}} = G_N/f'(R)$ :

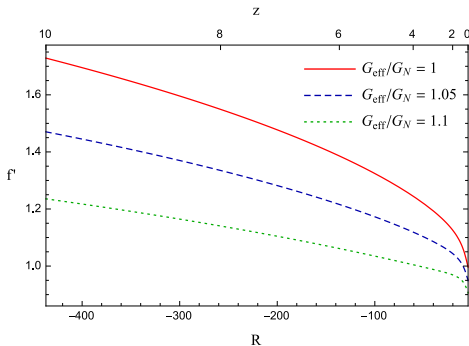
$$f'(R) > 0, \quad R \geq R_0 > 0$$

2. Constraints from Solar System tests, consistency with matter-dominated epoch and stability of cosmological perturbations:

$$f''(R) > 0, \quad R \geq R_0 > 0$$

3. Constraints from CMB observations:

$$f'(R) \rightarrow 1, \quad R \gg 1$$



Relaxing the assumption  $f'(R_0) = 1$ :

$$\begin{cases} f_0 = f'(R_0)(6H_0^2 + R_0) - 6H_0^2\Omega_{m0} \\ f_z|_{z=0} = f'(R_0) R_z|_{z=0} \end{cases}$$

[Capozziello, D'Agostino, Luongo,  
JCAP, 1805, 008 (2018)]  
[Capozziello, Cardone, Salzano,  
PRD 78, 063504 (2008)]

# $f(R)$ gravity in the Palatini formalism

- Action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m$$

- Varying the action with respect to  $g_{\mu\nu}$ :

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} , \quad F \equiv df/dR$$

- Varying the action with respect to  $\Gamma_{\mu\nu}^\alpha$ :

$$\nabla_\lambda (\sqrt{-g} F(R) g^{\mu\nu}) = 0$$

- Conformal metric:

$$h_{\mu\nu} \equiv F(R)g_{\mu\nu}$$

- Independent connection:

$$\Gamma_{\mu\nu}^\alpha = \tilde{\Gamma}_{\mu\nu}^\alpha + \frac{1}{2F} [2\delta_{(\mu}^\alpha \partial_{\nu)} F - g_{\mu\nu} g^{\alpha\sigma} \partial_\sigma F]$$

where  $\tilde{\Gamma}_{\mu\nu}^\alpha$  are the Christoffel symbols of  $g_{\mu\nu}$ .

- Ricci tensor:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{3}{2} \frac{(\nabla_\mu F)(\nabla_\nu F)}{F^2} - \frac{\nabla_\mu \nabla_\nu F}{F} - \frac{g_{\mu\nu}}{2} \frac{\square F}{F}$$

[Ferraris, Francaviglia, Volovich, CQG, 11, 1505 (1994)]

- Field equations:

$$G_{\mu\nu} = \frac{\kappa}{F} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( R - \frac{f}{F} \right) - \frac{3}{2F^2} \left[ (\nabla_\mu F)(\nabla_\nu F) - \frac{1}{2} g_{\mu\nu} (\nabla_\mu F)(\nabla^\mu F) \right] + \frac{1}{F} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) F$$

- Combining modified Friedmann equations:

$$\left( H + \frac{1}{2} \frac{\dot{F}}{F} \right)^2 = \frac{1}{6} \left[ \frac{\kappa(\rho + 3p)}{F} + \frac{f}{F} \right]$$

- Assuming pressureless matter and neglecting radiation:

$$\dot{R} = - \frac{3H(RF - 2f)}{F_R - F}$$

where  $F_R \equiv dF/dR = d^2 f/dR^2$ .

- Hubble expansion rate:

$$H^2 = \frac{1}{6F} \frac{2\kappa\rho_m + RF - f}{\left[ 1 - \frac{3}{2} \frac{F_R(RF - 2f)}{F(RF_R - F)} \right]^2}$$

# Reconstruction of the Palatini $f(R)$ action

- Relation between Ricci scalar of  $g_{\mu\nu}$  and Hubble parameter:

$$\tilde{R} = -6(\dot{H} + 2H^2) \implies \tilde{R}(z) = -6(1+z)H(z)H'(z) + 12H(z)^2$$

- Ricci scalar of the conformal metric:

$$R = 12H^2 - 6(1+z)H \left[ \frac{HF' + FH'}{F} \right] + 3(1+z)^2 \left[ \frac{2HF(HF'' + H'F') - H^2F'^2}{2F^2} \right]$$

- Matter energy density:  $\kappa\rho_m = 3H_0^2\Omega_{m0}(1+z)^3$

- Differential equation for  $F(z)$ :

$$F'' - \frac{3}{2} \frac{F'^2}{F} + \left( \frac{H'}{H} + \frac{2}{1+z} \right) F' - \frac{2H'}{H(1+z)} F + 3\Omega_{m0}(1+z) \left( \frac{H_0}{H} \right)^2 = 0$$

- Since  $G_{\text{eff}} = G_N/F$ , we impose:

$$F|_{z=0} = 1, \quad F'|_{z=0} = 0$$

- Initial condition for  $f(R)$ :

$$f_0 = 6H_0^2(\Omega_{m0} - 1) + R_0$$

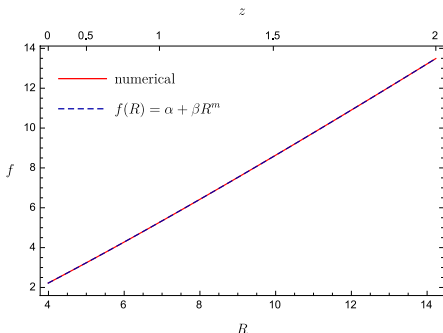
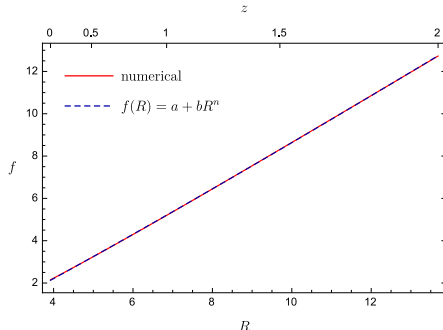
## 1) Using the (2,2) Padé approximation:

- Best analytical match:

$$(a, b, n) = (-3.866, 0.814, 1.073)$$

- $1\sigma$  bounds:

$$\begin{cases} a \in [-4.050, -2.909] \\ b \in [0.661, 0.931] \\ n \in [1.025, 1.124] \end{cases}$$



## 2) Using the (2,1) rational Chebyshev approximation:

- Best analytical match:

$$(\alpha, \beta, m) = (-1.332, 0.749, 1.124)$$

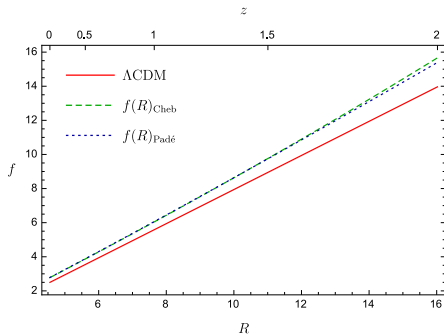
- $1\sigma$  bounds:

$$\begin{cases} \alpha \in [-1.481, -1.332] \\ \beta \in [0.749, 0.818] \\ m \in [1.096, 1.124] \end{cases}$$

[Capozziello, D'Agostino, Luongo, GRG, 51, 2 (2019)]



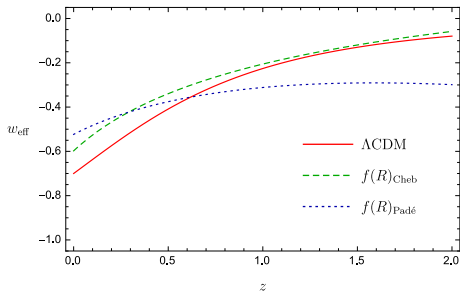
# Comparison with $\Lambda$ CDM



**Figure:** Comparison among the  $\Lambda$ CDM action and the  $f(R)$  reconstructed actions using the Padé and the rational Chebyshev approximations.

**Figure:** Comparison among the effective equation of state parameter for the  $\Lambda$ CDM model, the Padé and the rational Chebyshev reconstructions.

[Capozziello, D'Agostino, Luongo, GRG, 51, 2 (2019)]



# Conclusions and perspectives

- Cosmography is a procedure to reconstruct the Universe expansion in a model-independent way. The  $\Lambda$ CDM can be assumed as a "background" model [Capozziello, Nesseris, Perivolaropoulos, JCAP, 0712, 009 (2007)].
- Adopting rational polynomials in cosmography allows us to frame the late-time accelerated expansion of the Universe with an accuracy greater than the standard Taylor approach.
- Calibration orders of Padé polynomials and rational Chebyshev polynomials are compared with data: Chebyshev reduces systematics.
- MOG cosmography indicates departures from the standard  $\Lambda$ CDM model, showing that the equation of state is slightly evolving with respect to cosmic time.
- Cosmography disfavors Palatini  $f(R)$  models but it is in favor of metric  $f(R)$  models. Reconstructing functions frame out the Hubble shape.  $f(T)$  theories degenerate with respect to  $f(R)$ . MOG (in particular ETG) models seem favored with respect to unified DE models.
- **What next?** Extensions to very high  $z$ : High-redshift cosmography.
- **What next?** Comparisons with the Cosmic Microwave Background observations.
- **What next?** The issue of Hubble tension. Indication for New Physics?
- **What next?** Cosmography by gravitational waves?